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Impulse Control Maximum Principle: Theory and Applications

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Impulse Control Maximum Principle: Theory and Applications

Proefschrift

ter verkrijging van de graad van doctor aan Tilburg University op gezag van de rector magnificus, prof. dr. Ph. Eijlander, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op woensdag 27 februari 2013 om 14.15 uur door

Mohammed Chahim

geboren op 18 april 1985 te Fès, Marokko.

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ISBN 978 90 5668 344 3

Preface

As the title reveals, the topic of this dissertation is the Impulse Control Maximum Principle, which is part of optimal control theory. My first encounter with this topic was in the course Dynamic Capital Investment taught by Peter Kort and Jacob Engwerda. This course made me very curious about how to cope with non-static behavior in optimization models. This was one of the main reasons I decided to write a thesis with the topic optimal control theory. Not surprisingly, under the supervision of Peter Kort, and with Jacob Engwerda as second reader. The first steps in my scientific journey were made. This journey has almost come to an end. I would like to take the opportunity to express my gratitude to the people who have accompanied me and made it possible for me to reach this destination.

First, I would like to thank my promotores, Richard Hartl, Dick den Hertog and Peter Kort, for all the help I received and their faith in me.

I met Richard at one of his many visits to Tilburg University. During my Research Master I sent one of the first versions of my Research Master thesis to Richard. He carefully made comments on it and later joined as a second reader. Richard was always eager to discuss my research during our frequent encounters.

During the course orientation OR/MS I was amazed by a very enthusiastic professor, whose (OR related) anecdotes I will never forget. I also enjoyed our conversations about life (especially the purpose of life) and political discussions. Dick, thank you for your guidance. Thank you, for being honest and teaching me to be very critical.

Besides this dissertation, Peter also supervised my Master and Research Master thesis. He was the first one to introduce me into the world of control theory. He was always willing to share his wide spread network, and he made it possible for me to visit the optimal control research group in Vienna (ORCOS) on several occasions. He taught me that doing research requires good brain and good ideas, but also plain hard work.

Furthermore, I am grateful to Jacob Engwerda, Hans Schumacher, Dolf Talman and George Zaccour for joining Richard, Dick, and Peter in my dissertation committee. Thank you for taking the time to read my dissertation and for offering many suggestions for improvement.

It was a pleasure to work with Ruud Brekelmans which resulted in a paper presented in Chapter 3. Special thanks goes to Dieter Grass, who I visited several times in Vienna. These visits turned out to be very fruitful (as it resulted in two research papers presented in Chapter 4 and 5). Thanks for inviting me at your home in Purkersdorf Sanatorium and the pasta you made when we took a break from doing research. It think my programming skills improved every time I talked with you, whether this was in Vienna or via Skype. I would also like to thank the staff and students of ORCOS at the Vienna University of Technology for their hospitality.

Under the guidance of Peter and Dick I wrote a research proposal that was awarded a Mosaic grant from the Netherlands Organization for Scientific Research (NWO). I would like to express my gratitude to the organization, since they made it possible to write this dissertation. This research was supported under project number: 017.005.047.

Besides doing research, I had the pleasure to cooperate with Carol, Elleke, Feico, Gert, Hans B., Jacob, Jochem, Marieke, Marloes, Thijs, and Willem in various courses at Tilburg University. Special mention go to Elleke, Marieke, and Jacob for all the teaching related tips and feedback I received from them. This finally resulted in receiving an "Excellent teaching award" in 2012 for the course "Statistics for HBO".

I would like to thank CentER Graduate School and the department of Econometrics and Operations Research for hosting a cheerful work environment. First, I would like to thank Elleke, with whom I shared room K513. Thanks for tolerating all the people that visited me during that time. For me it was very delightful to have you as a roommate. Every time I had a question, or got stuck you were there to listen. I thank you and Vishwa for the many LaTeX and MATLAB tips and for providing me with your LaTeX dissertation framework.

Vishwa, thanks for the many discussions we had (about career, research and life), but far most for being a good friend. Many thanks go to Salima for the many pleasant conversations we had, for always making time when I entered your office and for listening to (and laughing at) my anecdotes. I also should mention John here, Salima's roommate, as he many times took part in our discussions (although sometimes I felt it was against his will).

Now, it would be the time to thank all (other) people at Tilburg University. Among them are Anja, Barış, Bas, Bertrand, Christian, Edwin, Gerwald, Hans R., Henk, Herbert, Jalal, Jarda, Kuno, Luc, Martin, Michele, Miguel, Mohammadi, Moazzam, Ning, Özer, Peter B., Ramon, Rene, Roy, Ruud, Takamasa, Tural. A special mention goes to İhsan whose office I always visited whenever I needed a break. He made the trip to Germany, to attend the SIAM conference on optimization, very amusing. I am also thankful to the departmental secretaries, Anja, Annemiek, Heidi, and Korine, who were always there for me.

I thank my fellow board members from student association Menara for organizing many discussions, workshops and study trips. I express my gratitude to Bauke, Bilal, Tarik and Tom, with whom I organized most activities with, for being good friends. Special thanks goes to Henri Geerts for all the activities we organized together, for all the discussions about the Netherlands, our multicultural society and many other subjects. Henry, thanks for becoming such a good friend.

I am indebted to the people of the PvdA (the Dutch Labour Party) in Helmond, with whom I intensively worked with in the last seven years and always had interest for my work in Tilburg. I want to thank all my friends and former classmates. Special mention for Bram and Paul, with whom I spent a lot of time while working on assignments and studying in the library. Thanks go to Mostapha and Mustafa for being very close friends.

Finally, I would like to show my appreciation to the people closest to me. To my brothers and sister, for creating such a competitive environment at home. This has helped shape me as the person I am today. Thank you so much for all your encouragements. I am glad to find two of my brothers Morad and Anoir willing to be my paranymphs and stand by my side during the defense of my dissertation. To my parents, Omar and Yamna: I will eternally be grateful for everything you have done for me. To you *mama* and *baba* I dedicate this book. Last, but certainly not least, a final words of thanks and sincere gratitude to my wife Hoyem. She is always there for me when I need her. Thank you for your unconditional love and endless support.

Mohammed Chahim
Helmond/Tilburg, December 2012

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CHAPTER 1

Introduction

1.1 Impulse Control

The Mathematical Optimization Society defines optimization or mathematical optimization as follows: *"In a mathematical optimization (or programming) problem, one seeks to minimize or maximize a real function of real or integer variables, subject to constraints on the variables. The term mathematical optimization refers to the study of these problems: their mathematical properties, the development and implementation of algorithms to solve these problems, and the application of these algorithms to real world problems".* Mathematical optimization has found wide applications in many disciplines including economics, management, physics, and engineering. In this thesis we focus on deterministic optimization problems, where contrary to stochastic optimization the problem does not generate or use random variables.

For systems that evolve smoothly through time (i.e. dynamic systems), (continuous) dynamic optimization is a frequently used tool. Optimal control theory is the branch of mathematical optimization developed to find optimal control regimes for (continuous) dynamical systems. Let $x(t)$ denote the *state variable* of the system at time $t \in [0, T]$, where $T > 0$ stands for the time horizon of the problem or planning period. Examples for $x(t)$ could be the amount of natural resource at time t, the stock or inventory level at time t, or the capital stock at time t. In optimal control theory it is assumed that the system can be controlled using a so called *control variable*. Let the (real) variable $u(t)$ be a *control variable* of the system at time t. For example, $u(t)$ can be the amount of natural resource being used at time t, the production rate at time t, or the (continuous) maintenance at time t. The dynamics of the system is often represented by a *state equation* that specifies the rate of change in the state variable as a function of the state variable itself, the control variable and t:

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \tag{1.1}$$

where $\dot{x}(t)$ stands for the derivative of x with respect to t, i.e. $dx(t)/dt$, f is a given function representing the change in the state variable, and x_0 is the initial value of the state variable. When the initial value of the state and the optimal trajectory of the control variable $u(t)$ are known (*control trajectory*), we can determine the *state trajectory*, i.e. the value of the state variable $x(t)$ during the planning period. We choose the control variable such that the state and control trajectory maximize/minimize the objective function

$$\int_0^T F(x(t), u(t), t)dt + S(x(T), T), \tag{1.2}$$

where F is a function of $x(t)$, $u(t)$ and t, which stands for *profits/costs* and the function S is the *salvage value*, which is a function of the final value of the state at the end of the planing period, $x(T)$, and time T. Most of the time the control variable $u(t)$ is constrained by a set Ω_u of possible outcomes of the control variable $u(t)$, i.e. $u(t) \in \Omega_u$. The optimal control problem is given by

$$\begin{cases} \max_u \int_0^T F(x(t), u(t), t)dt + S(x(T), T), \\ \text{subject to} \\ \dot{x}(t) = f(x(t), u(t), t), \quad \text{for} \quad t \in [0, T], \\ x(0) = x_0, \quad u(t) \in \Omega_u. \end{cases} \tag{1.3}$$

Continuous dynamic optimization has its own limitation, however, namely that continuity is assumed, whereas in the real world shocks (i.e. abrupt changes) can occur that fundamentally change the dynamic of the system at particular points in time. For example, the entrance of a rival is a singular event that changes the ground rules for a monopolist. It could also occur that decisions affect the system such that the system does not change continuously but instantaneously. An example is a firm that decides to invest in new (more efficient) machines. Since we try to build mathematical models such that they represent an actual or real life situation as much as possible, theory is developed to analyze systems that allow these discontinuous changes to occur in the system.

Impulse Control theory allows discontinuity in the states controlled by so called *impulse control variables* v. At certain moments in time disruptive changes are allowed and the value of the state variable changes. Let τ_i ($i = 1, \ldots, N$, where N is a variable denoting the number of changes in the time interval $[0, T]$) represent the times at which the state variable encounters this discontinuous change given by

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i), v(\tau_i), \tau_i), \tag{1.4}$$

where g is a function of the state variable x at time τ_i, the impulse control variable v at time τ_i and τ_i, representing the (finite) change of the state variable at the jump instances. For example, $v(\tau)$ can represent the amount of natural resources that is drilled out for use and N the number of times drilled, $v(\tau)$ can denote the total production that is added to the inventory and N the number of times production is added to the inventory, or $v(\tau)$ could stand for the replacement of (parts of) the machine and N the number of times a (part of a) machine is replaced. Also, the impulse control variable $v(\tau)$ can be constrained by a set Ω_v. Usually, these impulse changes are associated with costs/profits concerning the system at these jump time instances. Let $G(x(\tau_i), v(\tau_i), \tau_i)$ denote the costs/profits associated with each change of the system caused by the impulse control variable at time τ_i. Then the objective (1.2) is changed into

$$\int_0^T F(x(t), u(t), t)dt + \sum_{i=1}^{N} G(x(\tau_i), v(\tau_i), \tau_i) + S(x(T), T). \tag{1.5}$$

Summing up, an Impulse Control problem can be presented as

$$\begin{cases} \max_{v,u,\tau,N} \int_0^T F(x(t), u(t), t)dt + \sum_{i=1}^N G(x(\tau_i), v(\tau_i), \tau_i) + S(x(T), T), \\ \text{subject to} \\ \dot{x}(t) = f(x(t), u(t), t), \quad x(0) = x_0, \quad \text{for} \quad t \neq \tau_i, \quad i = 1, \ldots, N, \\ x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i, v(\tau_i), \tau_i), \quad \text{for} \quad t = \tau_i, \quad i = 1, \ldots, N, \\ u(t) \in \Omega_u, \quad v(\tau_i) \in \Omega_v, \quad i \in \{1, \ldots, N\}. \end{cases} \tag{1.6}$$

This thesis focuses on deterministic Impulse Control problems that are analyzed by using the Impulse Control Maximum Principle. This implies that we do not consider stochastic Impulse Control problems. This excludes the theory of real options (see Dixit and Pindyck (1994)). Another alternative is the theory of (Hamilton-Jacobi-Bellman) quasi-variational inequalities (see Bensoussan and Lions (1984)). Although quasi-variational inequalities can also be applied to deterministic Impulse Control problems, it is mainly related to a stochastic framework (quasi-variational inequalities is quite comparable to the Hamilton-Jacobi-Bellman framework, i.e. as is stated in Bensoussan et al. (2006), under the framework of impulse control, the Hamilton-Jacobi-Bellman equation reduces to quasi-variational inequalities). In stochastic optimal control problems the state variables in the system are not known with certainty. Moreover, in stochastic optimal control it might not even be possible to measure the value of a state variable at a certain time. There is a lot of literature that deals with these types of problems and the methodology differs a lot from the deterministic case. Most of the literature that deals with stochastic optimal control problems use the

Hamilton-Jacobi-Bellman framework (see e.g. Sethi and Thompson (2006)) or (more general) dynamic programming (see e.g. Bertsekas (2005)).

As Impulse Control, Multi-Stage optimal control (see e.g. Grass et al. (2008)) is tailored to the sorts of situations that have fallen between the cracks with the traditional partition into static and dynamic optimization. In the last few years there has been rapidly growing interest in Multi-Stage optimal control. As mentioned before, like Impulse Control theory, this theory allows sudden discontinuous changes at discrete points in time. These changes can affect the state variables, but also the values of parameters, or even the equations describing the system itself. Unlike Impulse Control, Multi-Stage optimal control does not allow jumps in the state variables. In Impulse Control models found in the literature discontinuous changes in the states are allowed. This is in contrast with Multi-Stage optimal control. There each regime is defined by different dynamics and the main concern is to find the optimal switching times between the regimes. Here, a regime is understood as the specification of a system dynamics and an objective functional during a certain time interval. In this thesis we focus on models that allow the state variables to jump at some time points. Take, for example, dike maintenance, where the problem is to determine the optimal dike heightening scheme for a certain time horizon. Here, the dike is the state variable and its height is increased at certain time points. This model cannot be solved using Multi-Stage optimal control, because we have jumps in the state variable.

1.2 Impulse Control Maximum Principle

In 1977 Blaquière derives a Maximum Principle that provides necessary (and sufficient) optimality conditions to solve deterministic Impulse Control problems, the so called Impulse Control Maximum Principle see e.g. Blaquière (1977a; 1977b; 1979; 1985). In 1981 Seierstad derives necessary optimality conditions that coincide with those of Blaquière, see Seierstad (1981) and Seierstad and Sydsæter (1987). Another good source presenting the Impulse Control Maximum Principle is Sethi and Thompson (2006, pp. 324–330).

In Blaquière (1979) an example of an Impulse Control model is given that deals with the optimal maintenance and life time of machines. Here the firm has to decide when a certain machine has to be repaired (impulse control variable), and it has to determine the rate of maintenance expenses (ordinary control variable), so that the profit is maximized over the planning period. In Gaimon (1985; 1986) a firm determines the optimal times of impulse acquisition of automation and the change for manual output. The objective is to minimize costs associated with the deviation from a goal level of output. The purchase of automation is used to directly substitute for output resulting from manually operated equipment. Since automation is acquired at discrete times, the author solves the model using the Impulse Control Maximum Principle. In Luhmer (1986) the theory is applied to an inventory model and in Kort (1989) a dynamic model of the firm is designed in which capital stock jumps upward at discrete points in time that the firm invests. Rempala (1990) describes three different kinds of Impulse Control problems where the number of jumps is not fixed, i.e. there are N impulse moments. He distinguishes between

(a) the impulse times are fixed and the size of the impulse is free,
(b) the size of the jump is fixed and the impulse moments are free,
(c) both the size of the jump and the impulse moments are free.

In Rempala (1990) it is shown that cases (b) and (c) can be reduced to case (a), and finally gives a simple proof for the Impulse Control Maximum Principle in case (a).

The theory of optimal control has its origin in physics and engineering where discounting cash flows does not occur. For this reason, Blaquière (1977a; 1977b; 1979; 1985) derived his Maximum Principle considering Impulse Control problems without using current value Hamiltonians. Instead, he presents his Maximum Principle in the present value Hamiltonian form. In Chapter 2 of this thesis we transform Blaquière's present value analysis to a current value one and we include an overview of the literature that makes use of the Impulse Control Maximum Principle.

Besides approaches using the Impulse Control Maximum Principle, there exist many other approaches in the literature to solve Impulse Control problems. We have seen mixed integer nonlinear programming (see e.g. Brekelmans et al. (2012)), dynamic programming (see e.g. Eijgenraam et al. (2011) and/or Erdlenbruch et al. (2011)), value function approach (see e.g. Neuman and Costanza (1990)) and finally the gradient method approach (see e.g. Hou and Wong (2011)) as an alternative for the Impulse Control Maximum Principle. All approaches have advantages and disadvantages. We will come back to this in Section 1.3.

1.3 Approaches to Solve Impulse Control Problems

This thesis considers optimal control problems in which the state variable is allowed to jump at some time instant. Both the size of the jump and the time instant are taken as (additional) decision variables. Hence, we are dealing with problems as described by case (3) in Rempala (1990). The Impulse Control Maximum Principle provides necessary optimality conditions that can be used to find the optimal solution to problems defined by (1.6). In ordinary optimal control also sufficiency conditions are given that ensure that the candidate solution that is found using the necessary optimality conditions is the optimal solution. Remarkably, for the Impulse Control Maximum Principle we have not found any models in the literature that also fulfill the sufficiency conditions derived by Blaquière (more on this in Section 1.4).

As mentioned earlier, there are several ways to solve Impulse Control problems. In this section we present eight different approaches and their main characteristics. An overview of the approaches and their characteristics is presented in Table 1.1.

Forward algorithm (FA) Luhmer (1986) derives a forward algorithm that makes use of the Impulse Control Maximum Principle. It starts at $t = 0$ and uses the value of the costates (i.e. dual variable, in economics this is known as the shadow price) to initialize the algorithm. The forward algorithm has a drawback. Namely, the initial value of the costates is the choice variable, i.e. we have to guess the initial values for the costate variables. A wrong guess of the costate variables at the initial time results in a solution that does not satisfy the transversality conditions for the costate variables, which implies that the necessary optimality conditions are not satisfied. The algorithm returns the solution for the given input, it does not need discretization in time.

Backward algorithm (BA) Kort (1989) develops a backward algorithm that starts at the end of the planning period, i.e. $t = T$, and goes backwards in time. For the backward algorithm we start with choosing values for the state variables at time T, i.e. the state variable at time T is the choice variable. The resulting solution always satisfies the necessary optimality conditions, but here the problem is that the algorithm has to end up at the right value of the states at $t = 0$. In other words, with the backward algorithm one can apply the right necessary conditions to the wrong problem. In Chapter 3 of this thesis we describe and apply the backward algorithm to a real-life dike height optimization problem. As the forward algorithm, the backward algorithm returns the solution for the given input, it does not need discretization in time.

(multipoint) Boundary value problem (BVP) In Chapter 5 of this thesis we describe the (multipoint) boundary value problem. For the (multipoint) boundary value problem approach we do not need to specify inputs for the state or the costate (unlike the forward and backward algorithm). The idea behind this approach is that the canonical system (the set of differential equations) is solved such that all (boundary) conditions on the state(s) and costate(s) (e.g. initial conditions and transversality conditions) are satisfied. To find the solution of the problem we can apply a continuation strategy with respect to the time horizon T, i.e. T is our continuation variable. To initialize the algorithm, the problem is solved for $T = 0$. Given a solution for $T = 0$, T is increased (continued) during the continuation process whereas the conditions for possible jumps are monitored. If the conditions for a jump are satisfied, the boundary value problem is adapted to this situation. With this new solution the continuation is pursued. No discretization of time or state variables is needed.

Continuation algorithm (CA) The continuation algorithm is only applicable if the canonical system of the Impulse Control problem can be solved explicitly in $[0, T]$. The problem can be restated as a discrete dynamical system (without numerical discretization). As for the boundary value problem approach, to find the solution of the problem we can apply a continuation strategy with respect to the time horizon T, i.e. T is our continuation variable. To initialize the algorithm, the problem is solved for $T = 0$. Given a solution for $T = 0$, T is increased (continued) during the continuation process whereas the conditions for possible jumps are monitored. No discretization of time or state variables is needed.

Gradient algorithm (GA) If the dynamics (i.e. the canonical system) of an Impulse Control problem can be solved explicitly, the problem can be restated (without numerical discretization) as a finite dimensional problem/ discrete dynamical system. In this method the necessary optimality conditions are derived, which, of course, reproduce the necessary optimality conditions of the Impulse Control Maximum Principle. First, the derivatives (gradients) of the equality constraints and the derivatives of the objective are determined. This gives a set of equations and equal number of variables. For this method the number of jumps needs to be fixed beforehand in order to solve the problem.

Value function approach (VFA) In Neuman and Costanza (1990) the value function method is used to solve an Impulse Control problem. For the value function approach the number of jumps is fixed beforehand in order to solve the problem. For a fixed number of jumps the value function is defined and the optimum of this value function is derived. This problem is solved for different numbers of fixed jumps until the optimal number of jumps is found. Since we do not know the optimal number of jumps beforehand, this approach is only useful if the optimal number of jumps is small.

Dynamic programming (DP) Eijgenraam et al. (2011) solves the same problem as in Chapter 3 of this thesis using dynamic programming. Unlike the backward and forward algorithm, dynamic programming requires discretization in time and the states for each stage.

Mixed integer nonlinear programming (MINLP) The mixed integer non-linear programming approach seems very fruitful for high dimensional problems, see e.g. Brekelmans et al. (2012), where the nonhomogeneous dike optimization problem is analyzed. On the other hand, mixed integer nonlinear programming requires discretization of the planning period. For these discrete time points Brekelmans et al. (2012) introduce a $\{0,1\}$-variable, which takes the value 1 if a dike heightening occurs and the value 0 otherwise. The size of the dike heightening is then given by a continuous variable. Finally, this $\{0,1\}$-variable is also used to add fixed cost.

In this thesis only in Chapter 4 a higher dimensional Impulse Control problem occurs, i.e. an Impulse Control problem with more than one state variable. We there study the investment behavior of a firm that has two state variables. The first state variable is the capital stock, and the second state variable is the state of technology. We solve the model using the boundary value problem approach. Because the canonical system of the problem described in Chapter 4 is explicitly solvable, also the continuation algorithm could be used. In the literature we find another higher dimensional Impulse Control problem in Brekelmans et al. (2012) where a dike heightening problem for nonhomogenous dikes is studied. The problem is solved using a mixed integer nonlinear programming approach. Comparing (i.e. with respect to computation time etc.) the different approaches for higher dimensional Impulse Control problems remains a topic for future research. However, some first ideas can be given. For both the forward algorithm and the backward algorithm the solution is derived using a choice variable. For a higher dimensional choice variable it is much harder to find the optimal value. For dynamic programming it is known that it works really well for problems with low dimensions, since the numerical discretization of the problem increases exponentially when the problems increases in dimension. Finally, for both the value function approach and the gradient algorithm the number of first order

	Approach[a]							
	FA	BA	BVP	CA	GA	VFA	DP	MINLP
Discretize time[b]	O	O	O	O	O	O	X	X
Discretize state	O	X[c]	O	O	O	O	X[c]	X
Discretize costate	X[c]	O	O	O	O	O	O	O
Fixed number of jumps	O	O	O	O	X	X	O	O
Higher dimensional problems	O	O	R	R	R	R	O	X
Explicit solution canonical system	X	X	O	X	X	O	O	O

[a] Forward algorithm (FA), backward algorithm (BA), (multipoint) boundary value problem (BVP), continuation algorithm (CA), gradient algorithm (GA), value function approach (VFA), dynamic programming (DP), and mixed integer nonlinear programming (MINLP).

[b] We mark each approach by O, X, or R, meaning does not include this characteristic, includes this characteristic or more research is needed, respectively.

[c] BA only needs discretization of the state at the end of the time horizon (final stage), unlike dynamic programming where discretization is needed for time and for the heights (states) for each stage. Similar to the FA, the BA only needs discretization for the costate at the start of the time horizon (first stage).

Table 1.1 – Characteristics of different approaches

conditions increases. The problem for both still is how to determine the optimal number of jumps, since this needs to be fixed beforehand in order to find a solution.

1.4 Contribution and Outline

The contribution of this thesis is threefold. First, it extends the existing theory on Impulse Control by deriving the necessary optimality conditions in current value formulation and providing a transformation such that the Impulse Control Maximum Principle can be applied to problems having a fixed cost. Moreover, this thesis points out that meaningful problems found in the literature do not satisfy the sufficiency conditions. Second, in this thesis the Impulse Control Maximum Principle is applied to dike height optimization and product innovation. Third, it describes several algorithms that can be used to solve Impulse Control problems. In this subsection, we describe these contributions in more detail.

Theory

In this thesis we use Blaquière's Impulse Control Maximum Principle to present the necessary optimality conditions in current value formulation. As mentioned before, Blaquière (1977a; 1977b; 1979; 1985) derived his Maximum Principle considering Impulse Control problems without using the current value Hamiltonian. Instead, he presents his Maximum Principle in the present value Hamiltonian form. The main reason for this is that the theory of optimal control has its origin in physics and engineering where discounting cash flows does not occur. Furthermore, by reviewing the existing Impulse Control models in the literature, we point out that meaningful problems do not satisfy the sufficiency conditions. In particular, such problems either have a concave cost function, contain a fixed cost, or have a control-state interaction, which have in common that they each violate the concavity hypothesis used in the sufficiency theorem. The implication is that the corresponding problem may have multiple solutions that satisfy the necessary optimality conditions. Moreover, we show that problems with a fixed cost do not satisfy the conditions under which the necessary optimality conditions can be applied. However, we propose a transformation, which ensures that the application of the Impulse Control Maximum Principle still provides the optimal solution. Finally, we show that for some existing models in the literature no optimal solution exists.

Applications

In the literature there are not many applications of the Impulse Control Maximum Principle. In this thesis we analyze two different applications. The first concerns dike height optimization in the Netherlands. As far as we know it is one of the first real life application of the Impulse Control Maximum Principle.[1] We compare our analysis with the dynamic programming approach used in Eijgenraam et al. (2011) and show that the Impulse Control approach has some benefits over the dynamic programming approach. The second application deals with product innovations. We consider a firm that wants to undertake a product innovation where the number of innovations is endogenously determined by the model. We compare our results with a Multi-Stage optimal control approach derived in Grass et al. (2012) where the number of product innovations is predetermined before solving the model. One interesting fact is that we find that the firm does not invest when marginal profit (with respect to capital) becomes zero, but invests when marginal profit is negative. Finally, we solve the forest management problem described in Neuman and Costanza (1990). Since we do not need to fix the number of jumps and do not need to discretize time,

[1] The data is provided by Rijkswaterstaat, part of the Dutch ministry of Infrastructure and Environment.

we find a solution with a better objective value than Neuman and Costanza (1990) do.

Algorithms

In Chapter 3 of this thesis we describe and apply the backward algorithm to a real-life dike height optimization problem. We compare the results found with the backward algorithm to the dynamic programming approach used in Eijgenraam et al. (2011).

In Chapter 5 of this thesis we describe three different algorithms, from which two (as far as we know) are new in the literature. The first (new) algorithm considers an Impulse Control problem as a (multipoint) Boundary Value Problem and uses a continuation technique to solve it. The second (new) approach is the continuation algorithm that requires the canonical system to be solved explicitly. This reduces the infinite dimensional problem to a finite dimensional system of, in general, non-linear equations, without discretizing the problem. Finally, we present a gradient algorithm, where we reformulate the problem as a finite dimensional problem, which can be solved using some standard optimization techniques. This method has been developed in Hou and Wong (2011).

Outline of thesis

This thesis is based on four self contained independent chapters in the field of Impulse Control. There are some differences in notation between chapters.

In Chapter 2 (consists of Chahim et al. (2012c)) we consider a class of optimal control problems that allows jumps in the state variable. We present the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. Moreover, we present a transformation such that the Impulse Control Maximum Principle can be applied to problems having a fixed cost. Finally, we give an overview of several problems in the literature that apply the Impulse Control Maximum Principle, show that these problems do not satisfy the sufficiency conditions, and that some of these models have received incomplete treatment, in particular, some of them do not have an optimal solution.

In Chapter 3 (consists of Chahim et al. (2012a)) we apply the Impulse Control Maximum Principle to determine the optimal timing of dike heightenings as well as the corresponding optimal dike heightenings to protect against floods. This chapter presents one of the first real life applications of the Impulse Control Maximum Principle developed by Blaquière. We show that the proposed Impulse Control Maximum Principle approach performs better than dynamic programming with respect to computational time. This is caused by the fact that Impulse Control does not need discretization in time.

Chapter 4 (consists of Chahim et al. (2012b)) considers a firm that has the option to undertake product innovations. For each product innovation the firm has to install a new production plant. We find that investments are larger and occur in a later stage when more of the old capital stock needs to be scrapped. Moreover, we obtain that the firm's investments increase when the technology produces more profitable products. We see that the firm in the beginning of the planning period adopts new technologies faster as time proceeds, but later on the opposite happens. Furthermore, we find that the firm does not invest when marginal profit (with respect to capital) becomes zero, but investes when marginal profit is negative. Moreover, numerical experiments show that if the time it takes to double the efficiency of a technology is larger than the time it takes for the capital stock to depreciate to half of its original level, the firm undertakes an initial investment. Finally, we show that when demand decreases over time and when fixed investment cost is higher, then the firm invests less throughout the planning period, the time between two investments increases, and the first investment is delayed.

In Chapter 5 (consists of Grass and Chahim (2012)) we present three different algorithms that can be used to solve Impulse Control problems. The first algorithm considers the problem as a (multipoint) BVP. The second and third algorithm can be used if the canonical system of the problem can be solved explicitly. If that is the case, we can rewrite our Impulse Control problem as a discrete dynamical system (without numerical discretization) and solve it.

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CHAPTER 2

A Tutorial on the Deterministic Impulse Control Maximum Principle: Necessary and Sufficient Optimality Conditions

Abstract This chapter considers a class of optimal control problems that allows jumps in the state variable. We present the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. By reviewing the existing impulse control models in the literature, we point out that meaningful problems typically do not satisfy the sufficiency conditions. In particular, such problems either have a concave cost function, contain a fixed cost, or have a control-state interaction, which have in common that they each violate the concavity hypotheses used in the sufficiency theorem. The implication is that the corresponding problem in principle may have multiple solutions that satisfy the necessary optimality conditions. Moreover, we argue that problems with fixed cost do not satisfy the conditions under which the necessary optimality conditions can be applied. However, we design a transformation, which ensures that the application of the Impulse Control Maximum Principle still provides the optimal solution. Finally, we show that for some existing models in the literature no optimal solution exists.

2.1 Introduction

For many problems in the area of economics and operations research it is realistic to allow for jumps in the state variable. This chapter therefore considers optimal control models in which the time moment of these jumps as well as the size of the jumps are taken as (additional) decision variables. An example is Blaquière (1979) that deals with optimal maintenance and life time of machines. Here the firm has to decide when a certain machine has to be repaired (impulse control variable), and it has to determine the rate of maintenance expenses (ordinary control variable), so that the profit is maximized over the planning period. Blaquière (1977a; 1977b; 1979; 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so called Impulse Control Maximum Principle, that gives necessary (and sufficient) optimality conditions for solving such problems. Like Blaquière (1977a; 1977b; 1979; 1985), we consider a framework where the number of jumps is not restricted. This distinguishes our approach from, e.g., Liu et al. (1998), Augustin (2002, pp. 71–81) and Wu and Teo (2006), where the number of jumps is fixed (i.e. is taken as given).

This contribution focuses on deterministic impulse control problems that are analyzed by using the Impulse Control Maximum Principle. This implies that we do not consider stochastic impulse control problems. This excludes the theory of real options (see Dixit and Pindyck (1994)). Another alternative is the theory of Quasi-Variational Inequalities (QVI; see Bensoussan and Lions (1984)). Although QVI can also be applied to deterministic impulse control problems, it is mainly related to a stochastic framework. Other insightful QVI references include Bensoussan et al. (2006) on an inventory model employing an (s, S) policy and Øksendal and Sulem (2007).

The contribution of this chapter is fourfold. *First*, we give a correct formulation of the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. In this way we correct Feichtinger and Hartl (1986, Appendix 6) and Kort (1989, pp. 62–70). *Second*, by reviewing the existing impulse control models in the literature, we point out that meaningful problems do not satisfy the sufficiency conditions. In particular, such problems either have a concave cost function, contain a fixed cost, or have a control-state interaction that each violate the concavity hypotheses used in the sufficiency theorem. The implication of not satisfying the sufficiency conditions is that the corresponding problem in principle has multiple solutions that satisfy the necessary optimality conditions. In many cases, these multiple solutions can be represented by a so called tree-structure (see, e.g., Luhmer (1986), Kort (1989), Chahim et al. (2012)). *Third*, we show that several existing problems (Blaquière (1977a; 1977b; 1979), Kort (1989, pp. 62–70)) do not have an optimal solution. In particular, the solution of these problems contain an interval where a singular arc is approximated as much as possible by applying impulse chattering. *Fourth*, we observe that problems with a fixed cost have the property that the cost function is not a C^1 function i.e. it is not continuously differentiable. This implies that in principle, also the necessary optimality conditions do not hold, although they were applied in Luhmer (1986), Gaimon (1985; 1986a; 1986b) and Chahim et al. (2012) leading to correct solutions. This chapter provides a transformation, which ensures that the Impulse Control Maximum Principle can still be applied to problems with a fixed cost.

This chapter is organized as follows. Section 2.2 gives the general formulation of an impulse control model with discounting and presents the correct Impulse Control Maximum Principle in current value formulation (i.e. the necessary optimality conditions). Further we give sufficient conditions for optimality and provide a transformation which makes clear why the Impulse Control Maximum Principle can still be applied to problems with a fixed cost. In Section 2.3 we classify existing economic models involving impulse control, show why optimal solutions for some of them do not exist, and discuss the problems that arise with the sufficiency conditions. Section 2.4 contains our conclusion and further remarks.

2.2 Impulse Control

The theory of optimal control has its origin in physics and engineering where discounting cash flows does not occur. For this reason Blaquière (1977a; 1977b; 1979; 1985) derived his Maximum Principle considering impulse control problems without using current value Hamiltonians. Instead, he presents his Maximum Principle in the present value Hamiltonian form.

Section 2.2.1 transforms Blaquière present value analysis to a current value one, whereas Section 2.2.2 presents sufficiency conditions. Section 2.2.3 considers a subclass of impulse control problems, where the cost function contains a fixed cost.

2.2.1 Necessary Optimality Conditions

In this section we derive necessary optimality conditions for impulse control in current value Hamiltonian form. In doing so, we correct the necessary optimality conditions for impulse control given in Feichtinger and Hartl (1986, Appendix 6). Their theorem is based on the current value present value transformation. However, applying it here turns out to be not as straightforward as usual.

A general formulation of the impulse control problem with discounting is:

$$\max_{\boldsymbol{u},N,\tau,\boldsymbol{v}} \int_0^T e^{-rt} F(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt + \sum_{i=1}^{N} e^{-r\tau_i} G(\boldsymbol{x}(\tau_i^-), \boldsymbol{v}^i, \tau_i) + e^{-rT} S(\boldsymbol{x}(T^+)), \quad \text{(IC)}$$

subject to

$$\begin{aligned}
&\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), && \text{for } t \notin \{\tau_1, \ldots, \tau_N\},\\
&\boldsymbol{x}(\tau_i^+) - \boldsymbol{x}(\tau_i^-) = \boldsymbol{g}(\boldsymbol{x}(\tau_i^-), \boldsymbol{v}^i, \tau_i), && \text{for } i \in \{1, \ldots, N\},\\
&\boldsymbol{x}(t) \in \mathbb{R}^n, \quad \boldsymbol{u}(t) \in \Omega_{\boldsymbol{u}}, \quad \boldsymbol{v^i} \in \Omega_{\boldsymbol{v}}, \quad i \in \{1, \ldots, N\}, &&\\
&x(0^-) = x_0, \quad 0 \leq \tau_1 < \tau_2 < \ldots < \tau_N \leq T. &&
\end{aligned}$$

Here, $\boldsymbol{x}$ is the state variable, $\boldsymbol{u}$ is an ordinary control variable and $\boldsymbol{v}$ is the impulse control variable (and $v^i = v(\tau_i)$), where $\boldsymbol{x}$ and $\boldsymbol{u}$ are piecewise continuous functions of time[1]. Future cash flows are discounted at a constant rate r leading to the discount factor e^{-rt}. The number of jumps is denoted by N, τ_i is the time moment of the i-th jump, and $\boldsymbol{x}(\tau_i^-)$ and $\boldsymbol{x}(\tau_i^+)$ represent the left-hand and right-hand limit of $\boldsymbol{x}$ at τ_i, respectively (i.e. the state value just before a possible jump and immediately after a possible jump at time τ_i). The terminal time or horizon date of the system or process is denoted by $T > 0$, and T^+ stands for the time moment just after T. The profit of the system at time t is given by $F(\boldsymbol{x}, \boldsymbol{u}, t)$, $G(\boldsymbol{x}, \boldsymbol{v}, \tau)$ is the profit function associated with the i-th jump at τ_i, and $S(\boldsymbol{x}(T^+))$ is the salvage value, i.e. the total costs or profit associated with the system after time T (where $\boldsymbol{x}(T^+)$ stands for the state value immediately after a possible jump at time T). Finally, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$ describes the continuous change of the state variable over time between the jump points and $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}, \tau)$ is a function that represents the instantaneous (finite) change of the state variable when there is an impulse or jump at τ.

We assume that the domains $\Omega_{\boldsymbol{u}}$ and $\Omega_{\boldsymbol{v}}$ are bounded convex sets in $\mathbb{R}^n$. Further we impose that F, $\boldsymbol{f}$, $\boldsymbol{g}$ and G are continuously differentiable in $\boldsymbol{x}$ on $\mathbb{R}^n$ and $\boldsymbol{v}^i$ on Ω_v, $S(\boldsymbol{x}(T^+))$ is continuously differentiable in $\boldsymbol{x}(T^+)$ on $\mathbb{R}^n$, and that $\boldsymbol{g}$ and G are continuous in t. Finally, when there is no impulse or jump, i.e. $\boldsymbol{v}^i = 0$, we assume that

$$\boldsymbol{g}(\boldsymbol{x}, 0, t) = 0,$$

for all $\boldsymbol{x}$ and t. A typical solution for an Impulse Control problem is presented in Figure 2.1.

[1]Note that the necessary conditions also hold for measurable controls. We restrict ourselves to piecewise continuous functions since this is needed for sufficiency. Applications typically have piecewise continuous functions.

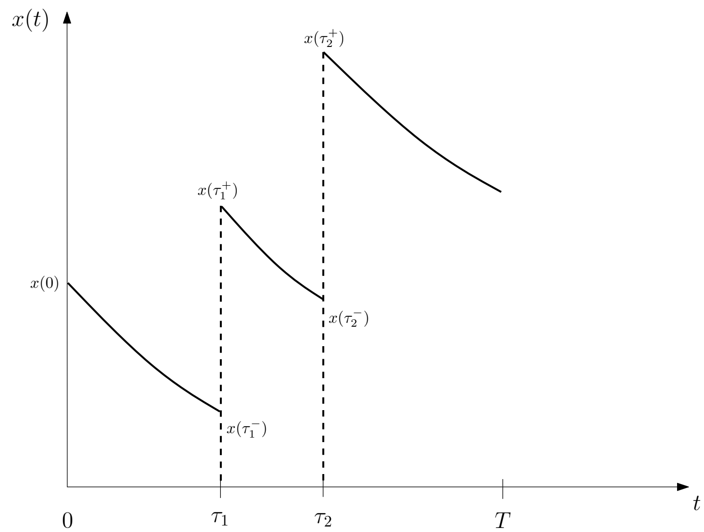


Figure 2.1 – Solution of Impulse Control system.

Let us define the present value Hamiltonian

$$\mathcal{H}am(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\mu}, t) = e^{-rt}F(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{\mu}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t),$$

and the present value Impulse Hamiltonian

$$\mathcal{IH}am(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\mu}, t) = e^{-rt}G(\boldsymbol{x}, \boldsymbol{v}, t) + \boldsymbol{\mu}\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}, t),$$

where $\boldsymbol{\mu}$ denotes the present value costate variable. The following theorem presents necessary optimality conditions associated with the impulse control problem defined in (IC).

Theorem 2.2.1 (Impulse Control Maximum Principle (present value))**.**
Let $(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), N, \tau_1^*, \dots, \tau_N, v^{1*}, \dots, v^{N*})$ be an optimal solution for the impulse control problem defined in (IC). Then there exists a piecewise continuous costate variable $\boldsymbol{\mu}(t)$ such that the following conditions hold:

$$\boldsymbol{u}^*(t) = \arg\max_{\boldsymbol{u}\in\Omega_{\boldsymbol{u}}} \mathcal{H}am(\boldsymbol{x}^*(t), \boldsymbol{u}, \boldsymbol{\mu}(t), t), \tag{2.1}$$

$$\dot{\boldsymbol{\mu}}(t) = -\frac{\partial \mathcal{H}am}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{\mu}(t), t), \quad \textit{for all} \quad t \neq \tau_i, \quad i = 1, \dots, N. \tag{2.2}$$

At the impulse or jump points, it holds that (i.e. at $t = \tau_i$, $i = 1, \dots, N$)

$$\frac{\partial \mathcal{IH}am}{\partial \boldsymbol{v}}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*)(\boldsymbol{v}^i - \boldsymbol{v}^{i*}) \leq 0, \quad \textit{for all} \quad \boldsymbol{v}^{\boldsymbol{i}} \in \Omega_{\boldsymbol{v}}, \tag{2.3}$$

$$\boldsymbol{\mu}(\tau_i^{*+}) - \boldsymbol{\mu}(\tau_i^{*-}) = -\frac{\partial \mathcal{IH}am}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*), \tag{2.4}$$

$$\mathcal{H}am(\boldsymbol{x}^*(\tau_i^{*+}), \boldsymbol{u}^*(\tau_i^{*+}), \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*)) - \mathcal{H}am(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{u}^*(\tau_i^{*-}), \boldsymbol{\mu}(\tau_i^{*-}), \tau_i^*)$$
$$-\frac{\partial \mathcal{IH}am}{\partial \tau}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*) \begin{cases} > 0 & \textit{if } \tau_i^* = 0 \\ = 0 & \textit{if } \tau_i^* \in (0, T) \\ < 0 & \textit{if } \tau_i^* = T. \end{cases} \tag{2.5}$$

For all points in time at which there is no jump, i.e. $t \neq \tau_i$ $(i = 1, \dots N)$, *it holds that*

$$\frac{\partial \mathcal{IH}am}{\partial \boldsymbol{v}}(\boldsymbol{x}^*(t), \boldsymbol{0}, \boldsymbol{\mu}(t), t)\boldsymbol{v} \leq 0, \quad \textit{for all} \quad \boldsymbol{v} \in \Omega_{\boldsymbol{v}}. \tag{2.6}$$

At the horizon date the transversality condition

$$\boldsymbol{\mu}(T^+) = e^{-rT}\frac{\partial S}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(T^+)), \tag{2.7}$$

holds, with $\boldsymbol{x}(T^+) = \boldsymbol{x}(T)$ *if there is no jump at time* T, *and* $\tau_1^* < \tau_2^* < \ldots < \tau_N^* \leq T$.

Proof: See Blaquière (1977a; 1985) or Rempala and Zabczyk (1988). ■

In Blaquière (1977a; 1985) it is assumed that the Impulse Hamiltonian is concave in $\boldsymbol{v}$. In this case (2.3) and (2.6) are replaced by

$$\boldsymbol{v}^{i*} = \arg\max_{\boldsymbol{v} \in \Omega_{\boldsymbol{v}}} \mathcal{IH}am(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^i, \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*), \quad \text{for} \quad i = 1, \dots, N,$$

and

$$\boldsymbol{0} = \arg\max_{\boldsymbol{v} \in \Omega_{\boldsymbol{v}}} \mathcal{IH}am(\boldsymbol{x}^*(t), \boldsymbol{v}, \boldsymbol{\mu}(t), t), \quad \text{for all} \quad \boldsymbol{v} \in \Omega_{\boldsymbol{v}},$$

respectively.

Next we determine the current value formulation of Theorem 1. By doing this we correct Feichtinger and Hartl (1986, Appendix 6), in which the current value version of condition (2.5) is wrongly stated. First, we define the current value Hamiltonian

$$Ham(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = F(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{\lambda}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t),$$

and the current value Impulse Hamiltonian

$$IHam(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\lambda}, t) = G(\boldsymbol{x}, \boldsymbol{v}, t) + \boldsymbol{\lambda}\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}, t),$$

with $\boldsymbol{\lambda}$ the current value costate variable. The following theorem presents necessary optimality conditions to solve the impulse control problem defined in (IC), based on the current value approach.

Theorem 2.2.2 (Impulse Control Maximum Principle (current value)).
Let $(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), N, \tau_1^*, \ldots, \tau_N, v^{1*}, \ldots, v^{N*})$ be an optimal solution for the impulse control problem defined in (IC). Then there exists a piecewise continuous costate variable $\boldsymbol{\lambda}(t)$ such that the following conditions hold:

$$\boldsymbol{u}^*(t) = \arg\max_{\boldsymbol{u}\in\Omega_{\boldsymbol{u}}} Ham(\boldsymbol{x}^*(t), \boldsymbol{u}, \boldsymbol{\lambda}(t), t), \tag{2.8}$$

$$\dot{\boldsymbol{\lambda}}(t) = r\boldsymbol{\lambda}(t) - \frac{\partial Ham}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t), t), \quad \textit{for all} \quad t \neq \tau_i, \quad i = 1, \ldots, N. \tag{2.9}$$

At the impulse or jump points, it holds that (i.e. at $t = \tau_i$, $i = 1, \ldots, N$)

$$\frac{\partial IHam}{\partial \boldsymbol{v}}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*)(\boldsymbol{v}^i - \boldsymbol{v}^{i*}) \leq 0, \quad \textit{for all} \quad \boldsymbol{v^i} \in \Omega_{\boldsymbol{v}}, \tag{2.10}$$

$$\boldsymbol{\lambda}(\tau_i^{*+}) - \boldsymbol{\lambda}(\tau_i^{*-}) = -\frac{\partial IHam}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(\tau_i^{*-}), v^{i*}, \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*), \tag{2.11}$$

$$\begin{aligned} &Ham(\boldsymbol{x}^*(\tau_i^{*+}), \boldsymbol{u}^*(\tau_i^{*+}), \boldsymbol{\lambda}(\tau_i^{*+}), \tau_i^*)) - Ham(\boldsymbol{x}^*(\tau_i^{*-}), u^*(\tau_i^{*-}), \boldsymbol{\lambda}(\tau_i^{*-}), \tau_i^*) \\ &\qquad - \left[\frac{\partial G}{\partial \tau}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \tau_i^*) - rG(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v}^{i*}, \tau_i^*)\right] \\ &\qquad -\boldsymbol{\lambda}(\tau_i^+)\frac{\partial \boldsymbol{g}}{\partial \tau}(\boldsymbol{x}(\tau_i^-), \boldsymbol{v^{i*}}, \tau_i) \begin{cases} > 0 & \textit{if } \tau_i^* = 0 \\ = 0 & \textit{if } \tau_i^* \in (0, T) \\ < 0 & \textit{if } \tau_i^* = T. \end{cases} \end{aligned} \tag{2.12}$$

For all points in time at which there is no jump, i.e. $t \neq \tau_i^*$ $(i = 1, \ldots N)$, it holds that:

$$\frac{\partial IHam}{\partial \boldsymbol{v}}(\boldsymbol{x}^*(t), \mathbf{0}, \boldsymbol{\lambda}(t), t)\boldsymbol{v} \leq 0, \quad \textit{for all} \quad \boldsymbol{v} \in \Omega_{\boldsymbol{v}}. \tag{2.13}$$

At the horizon date the transversality condition

$$\boldsymbol{\lambda}(T^+) = \frac{\partial S}{\partial \boldsymbol{x}}(\boldsymbol{x}^*(T^+)), \tag{2.14}$$

holds, with $\boldsymbol{x}(T^+) = \boldsymbol{x}(T)$ if there is no jump at time T, and $\tau_1^* < \tau_2^* < \ldots < \tau_N^* \leq T$.

Proof: The relation between present value and current value Hamiltonian, Impulse Hamiltonian and costate variables is given by

$$\mathcal{H}am(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\mu}, t) = e^{-rt} Ham(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\mu}, t),$$

$$\mathcal{IH}am(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\mu}, t) = e^{-rt} IHam(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{\mu}, t),$$

and

$$\boldsymbol{\mu}(t) = e^{-rt}\boldsymbol{\lambda}(t).$$

Under these transformations, conditions (2.8)-(2.11),(2.13) and (2.14) are equal to conditions (2.1)-(2.4),(2.6) and (2.7). In this proof we show that (2.12) is the current value equivalent of the analogous condition (2.5) derived by Blaquière (1977a; 1977b; 1979; 1985). From the definitions of *IHam* and $\mathcal{IH}am$ we obtain that

$$\begin{aligned} e^{-rt}IHam(\boldsymbol{x}(t), \boldsymbol{v^i}, \boldsymbol{\lambda}(t), t) &= e^{-rt}G(\boldsymbol{x}(t), \boldsymbol{v^i}, t) + e^{-rt}\boldsymbol{\lambda}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t) \\ &= e^{-rt}G(\boldsymbol{x}(t), \boldsymbol{v^i}, t) + \boldsymbol{\mu}(t)\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{v^i}, t) \\ &= \mathcal{IH}am(\boldsymbol{x}(t), \boldsymbol{v^i}, \boldsymbol{\mu}(t), t). \end{aligned}$$

Combining this with (2.5) we get for $\tau_i^* \in (0, T)$:

$$\mathcal{H}am(\boldsymbol{x}^*(\tau_i^{*+}), \boldsymbol{u}^*(\tau_i^{*+}), \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*)) - \mathcal{H}am(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{u}^*(\tau_i^{*-}), \boldsymbol{\mu}(\tau_i^{*-}), \tau_i^*) =$$
$$e^{-r\tau_i^*}\left(\frac{\partial G(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)}{\partial \tau} - rG(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)\right) + \boldsymbol{\mu}(\tau_i^{*+})\frac{\partial \boldsymbol{g}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)}{\partial \tau},$$

which implies that

$$\begin{aligned} Ham(\boldsymbol{x}^*(\tau_i^{*+}), \boldsymbol{u}^*(\tau_i^{*+}), \boldsymbol{\mu}(\tau_i^{*+}), \tau_i^*)) - Ham(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{u}^*(\tau_i^{*-}), \boldsymbol{\mu}(\tau_i^{*-}), \tau_i^*) \\ = e^{r\tau_i^*}\left(e^{-r\tau_i^*}\left(\frac{\partial G(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)}{\partial \tau} - rG(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)\right)\right) \\ + e^{r\tau_i^*}\boldsymbol{\mu}(\tau_i^{*+})\frac{\partial \boldsymbol{g}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)}{\partial \tau} \\ = \frac{\partial G(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)}{\partial \tau} - rG(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau) + \boldsymbol{\lambda}(\tau_i^{*+})\frac{\partial \boldsymbol{g}(\boldsymbol{x}^*(\tau_i^{*-}), \boldsymbol{v^{i*}}, \tau_i^*)}{\partial \tau}. \end{aligned}$$

This is condition (2.12) for $\tau_i^* \in (0, T)$. The other two cases, $\tau_i^* = 0$ and $\tau_i^* = T$, follow the same steps. ■

2.2.2 Sufficiency Conditions

The following theorem can be found in Seierstad and Sydsæter (1987, pp. 198–199).

Theorem 2.2.3 (Sufficient Conditions for Impulse Control)**.** *Let there be a feasible solution, $(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), N, \tau_1^*, \ldots, \tau_N, v^{1*}, \ldots, v^{N*})$, for the impulse control problem (IC) and a piecewise continuous costate trajectory, so that the necessary optimality conditions of Theorem 2.2.2 hold. When the maximized Hamiltonian function $Ham^0(\boldsymbol{x}, \boldsymbol{\lambda}, t) = \max_{\boldsymbol{u}} Ham(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t)$ is concave in $\boldsymbol{x}$ for all $(\boldsymbol{\lambda}, t)$, the IHam, concave in $(\boldsymbol{x}, \boldsymbol{v})$ for all t and $S(\boldsymbol{x})$ concave in $\boldsymbol{x}$, then that solution, $(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), N, \tau_1^*, \ldots, \tau_N, v^{1*}, \ldots, v^{N*})$, is optimal.*

For the proof of this theorem we refer to Theorem 1 in Seierstad (1981), which is equivalent to the theorem stated above. However, we will show in Section 2.3 that this result is not very useful since most (relevant) problems given in the literature do not fulfil these conditions.

2.2.3 Impulse Control: Including a Fixed Cost

When there is some *fixed cost* involved in the impulse cost function, the function G has a jump discontinuity at point $v^i = 0$. The implication is that G is not continuously differentiable. Consequently, strictly speaking the Impulse Control Maximum Principle cannot be applied. However, the Impulse Control Maximum Principle has been applied a few times while ignoring this continuity requirement (see, e.g., Luhmer (1986), Gaimon (1985; 1986a; 1986b) and Chahim et al. (2012)). In this section we show that by applying some transformation, a general fixed cost problem can be represented by a problem with continuous cost function so that still the necessary optimality conditions can be applied.

Reconsider the above general impulse control problem. For the remaining of this chapter we assume $\Omega_v = [0, \bar{v}]$ for some $\bar{v} > 0$ and $g(x, 0, t) = 0$ (see e.g., Blaquière (1977a; 1977b; 1979; 1985) and Seierstad and Sydsæter (1987)). Furthermore, the impulse cost function needs to be continuously differentiable. As said before, this is not the case in the specification where G is discontinuous because of a fixed cost term (for simplicity we delete the superscript i in v^i):

$$G(x, v, \tau) = \begin{cases} 0 & \text{for } v = 0 \\ K(\tau) + \alpha(v, \tau)v & \text{for } 0 < v \leq \bar{v}, \end{cases}$$

where $K(\tau) > 0$. Clearly G is lower semi-continuous.

The idea is to approximate the impulse cost function $K + \alpha v$ by a continuously differentiable one that assumes the same value for $v > \varepsilon$, where we let ε go to zero. A possible specification would be

$$G_\varepsilon(x, v, \tau) = \begin{cases} -\frac{K(\tau)}{\varepsilon^2}v^2 + (\frac{2K(\tau)}{\varepsilon} + \alpha(v, \tau))v & \text{for } v \in [0, \varepsilon] \\ K(\tau) + \alpha(v, \tau)v & \text{for } \varepsilon < v \leq \bar{v}. \end{cases}$$

Letting ε tend to zero it follows that G_ε approaches G. Other specifications of $G_\varepsilon(x, v, \tau)$ are also possible, but the common property is that $\lim\limits_{\varepsilon \to 0} \frac{\partial}{\partial v} G_\varepsilon(x, 0, \tau) = \infty$. The argument is that the optimal solution of a problem with cost G will never have "very small" jumps because of the fixed costs. Then, for ε small enough, G_ε will always generate the same cost as G and the optimal solutions will be the same. Hence, the necessary optimality conditions still hold for G with fixed cost. The following lemma and proposition formalize these statements.

Lemma 2.2.1. *Let* $0 < \varepsilon_1 < \varepsilon_0$ *and let* $(x_\varepsilon, u_\varepsilon, v_\varepsilon)$ *(for simplicity we omit* τ *and* N*) be an optimal solution of the problem with cost function* G_ε*, while* (x^*, u^*, v^*) *is an optimal solution of problem (IC). Furthermore, we denote by* $J(x, u, v)$ *the value of the objective function of the original problem evaluated at* (x, u, v)*, and by* $J_\varepsilon(x, u, v)$ *the value of the objective function of the approximated problem with cost function* G_ε *evaluated at* (x, u, v)*. Then*

$$J(x, u, v) \leq J_{\varepsilon_1}(x, u, v) \leq J_{\varepsilon_0}(x, u, v), \tag{2.15}$$

and

$$J(x^*, u^*, v^*) \leq J_{\varepsilon_1}(x_{\varepsilon_1}, u_{\varepsilon_1}, v_{\varepsilon_1}) \leq J_{\varepsilon_0}(x_{\varepsilon_0}, u_{\varepsilon_0}, v_{\varepsilon_0}). \tag{2.16}$$

Proof: The first result (2.15) follows directly from $G_{\varepsilon_0} \leq G_{\varepsilon_1} \leq G$, whereas (2.16) follows from (2.15) and

$$J_{\varepsilon_1}(x_{\varepsilon_1}, u_{\varepsilon_1}, v_{\varepsilon_1}) \leq J_{\varepsilon_0}(x_{\varepsilon_1}, u_{\varepsilon_1}, v_{\varepsilon_1}) \leq J_{\varepsilon_0}(x_{\varepsilon_0}, u_{\varepsilon_0}, v_{\varepsilon_0}).$$

■

Proposition 2.2.1. *Let* (x^*, u^*, v^*) *(for simplicity we omit* τ *and* N*) be an optimal solution of problem (IC). Then the Impulse Control Maximum Principle provides necessary optimality conditions, even though the model function* G *is not continuous. More precisely, if the optimal solution is unique, it satisfies these necessary optimality conditions. Otherwise there is at least one optimal solution for which this holds.*

Proof: Let ε_0 be some small positive number and let $(x_{\varepsilon_0}, u_{\varepsilon_0}, v_{\varepsilon_0})$ be an optimal solution of the problem with cost function G_{ε_0}, which thus satisfies the necessary optimality conditions. Let further $v^i_{\varepsilon_0}$ be the smallest jump parameter in this optimal solution. If $v^i_{\varepsilon_0} \geq \varepsilon_0$, the proposition automatically holds. If $v^i_{\varepsilon_0} < \varepsilon_0$, choose a lower ε_0, and check again whether $v^i_{\varepsilon_0} \geq \varepsilon_0$. If yes we are done, if not continue this procedure. ■

2.3 Classification of Existing Operations Research Models Involving Impulse Control

This section classifies existing operations research impulse control problems found in the literature. When considering impulse control problems in an operations research context, a common feature is discounting. The resulting general impulse control problem (where for reasons of exposition both the state and impulse control are one dimensional) can be represented by

$$\max_{u,v,\tau,N} \int_0^T e^{-rt} F(x(t), u(t), t)dt + \sum_{i=1}^{N} e^{-r\tau_i} G(x(\tau_i^-), v_i, \tau_i) + e^{-rT} S(x(T^+)), \quad (2.17)$$

subject to

$$\begin{array}{ll} \dot{x}(t) = f(x(t), u(t), t), & \text{for } t \notin \{\tau_1, \ldots, \tau_N\}, \\ x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v^i, \tau_i), & \text{for } i \in \{1, \ldots, N\}, \\ x \in \mathbb{R}, \quad u \in \Omega_u, \quad v^i \in \Omega_v, \quad x(0^-) = x_0, & 0 \le \tau_1 < \tau_2 > \ldots, < \tau_N \le T \ . \end{array}$$

The objective is typically to maximize profit or minimize cost. We distinguish between

- linear impulse control problem, i.e. a problem where the impulse control variable occurs linearly in the Impulse Hamiltonian, and *no* continuous control present (*Case A*);
- linear impulse control problem and continuous control present (*Case B*);
- *non*-linear impulse control problem and *no* continuous control present (*Case C*);
- *non*-linear impulse control problem and continuous control present (*Case D*).

In the *linear* impulse control case where *no continuous* control u is present (*Case A*), a typical solution would be to reach some kind of singular arc by applying impulse control, but, if the state equation contains some decay term (for instance $\delta K(t)$ with δ the depreciation rate and $K(t)$ the capital stock), then it might be formally impossible to stay there. One has to use some kind of impulse chattering, i.e. an infinitely large number of impulses of infinitely small size. We elaborate on this when discussing the model by Blaquière (1977a; 1977b) in Section 2.3.1.

In the *linear* impulse control case where *also a continuous* control u is present (*Case B*) and both the ordinary control and the impulse control go into the same direction i.e. increase or decrease the state, the two controls (i.e. the ordinary and impulse control) are in some sense substitutes to each other. Then one can distinguish the following cases

1. Continuous control u and impulse control v have the same monetary effect (e.g. cost or profit). An example is the model by Seierstad and Sydsæter (1987, pp. 199–202) where just the impulse control is used to sell the complete stock of the resource at the best point in time. It is a non-autonomous model were the two controls appear in the model in the same way and are substitutes. The jump occurs at one time instant and in that sense this model is comparable to a model that has the most rapid approach path (MRAP) property (see e.g. Hartl and Feichtinger (1987)), where the singular arc is reached by applying impulse control at one point of time (usually the initial time point), followed by a singular arc which is maintained using the continuous control. The same analysis hold for the model by Seierstad and Sydsæter (1987, pp. 202–206). Other existing optimal control models having this MRAP property are, e.g., Jorgenson (1963; 1967), and Sethi (1973). These kinds of models are not considered in this chapter any further.

2. The impulse control has a higher cost. An example is the model by Blaquière (1979)(see Section 2.3.2), where, for suitable values of $x(0)$, only the continuous control is used to apply preventive maintenance for the machine but no impulse control to repair or upgrade the machine. If $x(0)$ is very low an impulse jump occurs at the initial time (MRAP-property), after which preventive maintenance is applied.

3. The impulse control has a lower cost. An example would be the model by Blaquière (1979)(see Section 2.3.2), with modified parameters so that repair is more attractive than preventive maintenance. Then one would not do preventive maintenance but only repair during the planning period. This will lead to an impulse chattering solution. We demonstrate in Section 2.3.1 that in such cases no optimal solution exists.

In some sense, these results are trivial, i.e. there is no interesting combination of the two types of control. Such interesting cases occur when there is some *fixed cost* involved in the impulse cost function. In the *non-linear* impulse control case where *no continuous* control u is present (*Case C*) this fixed cost in the impulse cost function often occurs, examples are e.g. Luhmer (1986) and Chahim et al. (2012). In Kort (1989) a model is given that analyzes the behavior of a firm under a concave adjustment cost function where impulse control is applied. However, in Section 2.3.5 we demonstrate that an optimal impulse control solution does not exist!

In the literature no problems exist dealing with the non-linear impulse control case where the continuous control u is present (*Case D*). This is different in the literature on stochastic impulse control, where, e.g., Bensoussan and Lions (1984, Chapter 1, Section 4) discuss an inventory problem with continuous production and impulse ordering of goods. However, as said before, this chapter restricts itself to a deterministic impulse control framework, and, since "*Case D* problems" do not occur in this literature, we will not consider this case any further.

In the next sections we will discuss several (relevant) problems, check whether the sufficiency conditions of Theorem 2.2.3 hold, and describe the nature of the solutions. In particular we prove that in the roadside inn problem (Section 2.3.1), in one scenario of the maintenance problem in Section 2.3.2, and in the investment problem of Section 2.3.5 no optimal solution exists. These problems have in common that "impulse chattering" occurs on a time interval with positive length. This impulse chattering is called "micro-impulse policy" in Erdlenbruch et al. (2011). On the other hand, for problems in Section 2.3.3 (Luhmer (1986)), Section 2.3.4 (Gaimon (1985; 1986a; 1986b)) and Section 2.3.6 (Chahim et al. (2012)) an algorithm is designed that employs the necessary optimality conditions to find all candidate solutions for optimality, as is shown in Luhmer (1986) (see also Kort (1989) and Chahim et al. (2012). Out of these candidate solutions, we can simply select the one with the highest objective value. Provided that an optimal solution exists, this is then for sure the optimal solution.

2.3.1 Maximizing the Profit of a Roadside Inn (*Case A*)

In Blaquière (1977a; 1977b) an example is given that deals with maximizing the profit of the owner of a roadside inn. The owner attracts more customers if he repaints the inn. The following model is given:

$$W(T) = \max_{v,N} A \int_0^T x(t)dt - C \sum_{i=1}^{N} v^i, \tag{2.18}$$

subject to

$$\begin{aligned}
&\dot{x}(t) = -kx(t), && \text{for } t \notin \{\tau_1, \dots, \tau_N\}, \\
&x(\tau_i^+) - x(\tau_i^-) = v^i(1 - x(\tau_i^-)), && \text{for } i \in \{1, \dots, N\}, \\
&x(t) \in [0,1], \quad v^i \in [0,1], \quad x(0^-) = x_0, && 0 \le \tau_1 < \tau_2 > \dots, < \tau_N \le T \ ,
\end{aligned}$$

where N is the number of times the inn is (re)painted, $C > 0$, the marginal cost of each (re)paint job, A a strictly positive constant, and v^i denotes the part of the roadside inn that needs to be repainted, where $v^i = 1$ denotes a full repaint. The appearance of the roadside inn is denoted by x. It is assumed that $0 \le x \le 1$, and each time the inn is repainted the index of appearances of the inn x undergoes an upward jump from its previous value $x(\tau_i^-)$. Between (re)painting x decays as given above, with the depreciation rate k being a positive constant. Furthermore, we assume that after the planning period the inn will not be used (i.e. the salvage value is set to zero). In Sethi and Thompson (2006, pp. 324–330) this problem has been reinterpreted as "The Oil Driller's Problem".

The Hamiltonian and Impulse Hamiltonian in short hand notation are

$$\mathcal{H}am(x,\mu) = Ax + \mu(-kx),$$

$$\mathcal{IH}am(x,v,\mu) = v(-C) + \mu v(1-x) = v(-C + \mu(1-x)).$$

Both the impulse control variable and state variable are linear in $\mathcal{IH}am$ and $\mathcal{H}am$. Due to the interaction term between the impulse control variable and the state variable in the Impulse Hamiltonian, $\mathcal{IH}am$ is not concave in (x, v^i) jointly, so that the necessary optimality conditions are not sufficient.

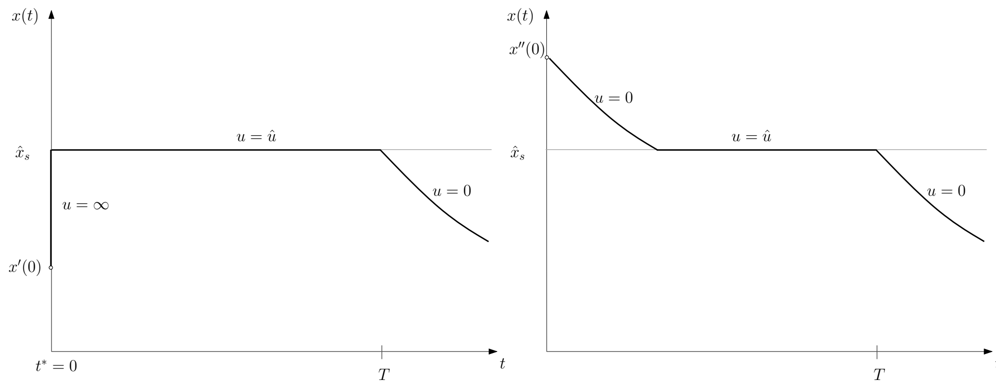
To solve the above stated model we first consider the continuous version of this problem (i.e. the problem where the impulse control v^i is replaced by a continuous control u):

$$\max_u \quad W(T) = \int_0^T (Ax(t) - Cu(t))dt, \tag{2.19}$$

subject to

$$\dot{x}(t) = -kx(t) + u(t)(1-x(t)),$$
$$x(t) \in \mathbb{R}, \quad u(t) \in [0,\infty) \quad x(0) = x_0.$$

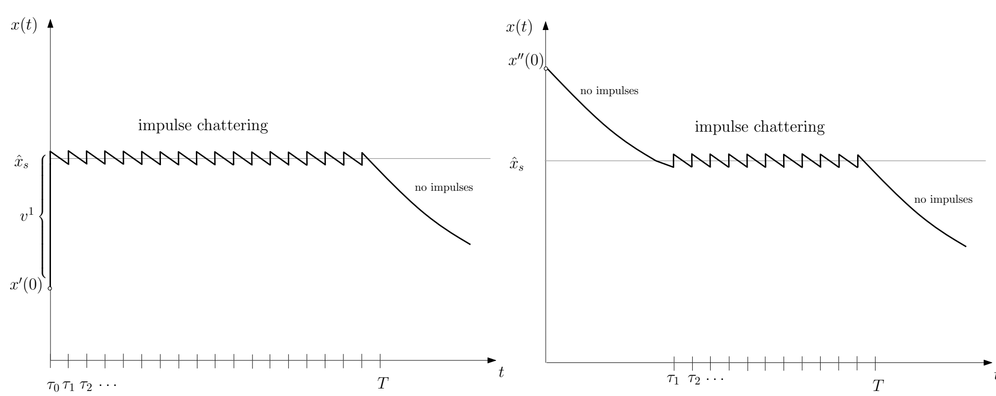
We can identify this model as the *Vidale-Wolfe* advertising model discussed in Sethi (1973). The solution for this model is given in Figure 2.2. If the initial value of $x(0)$ is lower than the singular arc value of $x(t)$ (i.e. $\hat{x}_s$) at $t^* = 0$, we set the control $u = \infty$ so that the singular arc is reached immediately (MRAP property). If the initial value of $x(0)$ is higher than $\hat{x}_s$ the control $u = 0$ is applied until x has reached $\hat{x}_s$. At the singular arc the control is set at $u = \hat{u}_s = k\hat{x}_s/(1-\hat{x}_s)$, so that $x(t)$ is kept constant at the level $\hat{x}_s$. At the final planning period the control is equal to zero, since the remaining time period is too short to defray the cost uC. To solve the Blaquière impulse control model, we need to approximate the *Vidale-Wolfe* advertising model as much as possible. This is straightforward for the solution part where $u = 0$ (then simply put $v^i = 0$) or where $u = \infty$. In the latter case apply an initial impulse control jump, where $v^1 = \hat{x}_s - x'(0)$. On the singular arc we divide the interval $[t_{sa}, T]$ (with t_{sa} the time the singular arc is reached) in l parts of equal length and



(a) $x'(0) < \hat{x}_s$. **(b)** $x''(0) > \hat{x}_s$.

Figure 2.2 – Vidale-Wolfe model solution.

set within each interval first $v^i = \bar{v}$ (where $\bar{v}$ is such that $\tilde{x} + \bar{v} - \hat{x}_s = \hat{x}_s - \tilde{x}$ with $\tilde{x} = x(\tau_1^-) = \ldots = x(\tau_N^-)$) and then $v^i = 0$. In this way we create a "saw-toothed" shape around the singular arc. This control policy is shown in Figure 2.3 and is the impulse control equivalent of chattering control (see e.g. Feichtinger and Hartl (1986, pp. 78–81) or Kort (1989, pp. 62–70)). It is important to note that for each given "saw-toothed" solution, a better solution is available by increasing l and decreasing $\bar{v}$. We conclude that an optimal solution does not exist. This observation cannot be found in Blaquière (1977a; 1977b), or in Sethi and Thompson (2006, pp. 324–330).



(a) $x'(0) < \hat{x}_s$. **(b)** $x''(0) > \hat{x}_s$.

Figure 2.3 – Blaquière (1977) model solution with impulse chattering.

2.3.2 Optimal Maintenance of Machines (*Case B*)

The following problem is taken from Blaquière (1979) and is also extensively analyzed in Sethi and Thompson (2006, pp. 331–337). This example deals with the optimal maintenance of machines:

$$W(T) = \max_{v,u,\tau,N} \int_0^T (Ax(t) - u(t))dt - \sum_{i=1}^{N} v^i (C - Kx(\tau_i^-)), \tag{2.20}$$

subject to

$$\begin{aligned}
&\dot{x}(t) = -kx(t) + mu(t), && \text{for } t \notin \{\tau_1, \ldots, \tau_N\}, \\
&x(\tau_i^+) - x(\tau_i^-) = v^i(1 - x(\tau_i^-)), && \text{for } i \in \{1, \ldots, N\}, \\
&x \in [0,1], \quad v^i \in [0,1], \quad u(t) \in [0, \bar{u}], \quad x(0^-) = x_0, \quad 0 \le \tau_1 < \tau_2 > \ldots, < \tau_N \le T,
\end{aligned}$$

where N is the number of times the machines is repaired, $C - Kx(\tau_i)$, $i = 1, ..., N$, the marginal unit cost of each repair, and A a strictly positive constant. It is assumed that $0 \le x \le 1$, and each time the machine is repaired (where full repair, i.e. $v^i = 1$ stands for replacing the machine with a new one) the index of appearances of the machine, x, undergoes an upward jump starting from its previous value $x(\tau_i^-)$. Between repairs x decays as given above, with k and m positive constants. The rate of maintenance expenses is denoted by u (i.e. the continuous control). Moreover it is assumed that the cost of a repair is of the form $v^i(C - Kx)$, where C and K are strictly positive constants. Furthermore, we assume that after the planning period the machine will not be used (i.e. the salvage value is set to zero). The Hamiltonian and Impulse Hamiltonian are

$$\mathcal{H}am(x, u, \mu) = Ax - u + \mu(-kx + mu),$$

$$\mathcal{IH}am(x, v, \mu) = v(Kx - C) + \mu v(1 - x) = v(Kx - C + \mu(1 - x)).$$

Both the impulse control variable and state variable are linear in $\mathcal{IH}am$ and $\mathcal{H}am$. Due to the interaction term between the impulse control variable and the state variable in the Impulse Hamiltonian the necessary optimality conditions are not sufficient, since $\mathcal{IH}am$ is not concave in (x, v^i). Because the necessary optimality conditions are not sufficient we know that multiple candidate solutions can occur for this problem. Here we will distinguish between two cases:

- The impulse control (repair) has a higher cost than the continuous control (preventive maintenance). When $x(0)$ is sufficiently large, only the continuous control is used to do preventive maintenance for the machine, so no impulse control is applied to repair or upgrade the machine. In this case the coefficients satisfy $mK \leq 1 < mC < \frac{mA}{k}$. When $x(0)$ is very low, besides preventive maintenance, an impulse jump occurs at the initial time and in that sense this model is comparable to a model that has the most rapid approach path (MRAP) property. For the analysis of this case we refer to Blaquière (1979).

- The impulse control (repair) has lower cost than the continuous control (preventive maintenance). Then one would not do preventive maintenance but repair during the planning period. This results in impulse chattering analogous to the Blaquière (1977a; 1977b) model in Section 2.3.1. Hence, for this case no optimal solution exists.

2.3.3 Minimizing Inventory Cost (*Case C*)

Luhmer (1986) applies the Impulse Control Maximum Principle to solve an inventory problem. The following model is presented:

$$C(T) = \min_{v,\tau,N} \int_0^T h(I(t),t)e^{-rt}dt + \sum_{i=1}^{N} \Big(p(v^i,\tau_i)v^i + C(\tau_i)\Big)e^{-r\tau_i} - S(I(T))e^{-rT}, \qquad (2.21)$$

subject to

$$\begin{aligned}
&\dot{I}(t) = -d(t) - g(I(t),t), && \text{for } t \notin \{\tau_1,\ldots,\tau_N\},\\
&I(\tau_i^+) - I(\tau_i^-) = v^i, && \text{for } i \in \{1,\ldots,N\},\\
&I(t) \in \mathbb{R}^+, \quad v^i \in (0,\infty), && I(0) = I_0, \quad I(T) = I_e, \quad 0 \leq \tau_1 < \tau_2 > \ldots, < \tau_N \leq T,
\end{aligned}$$

where h denotes the holding or shortage cost and $I(t)$ the inventory level at time t. $I(t)$ decreases over time by the demand rate $d(t)$ and leakage losses $g(I(t),t)$. At any time instance τ_i the inventory is increased by a quantity v^i and the unit ordering costs are given by $p(v^i,\tau_i)$. An order of size v^i at time τ_i results in a variable cost of $(p(v^i,\tau_i)v^i$ plus a fixed ordering cost $C(\tau_i)$. At the end of the planning period a scrap value for inventory I_e is left over, which is denoted by $S(I(T))$. Finally, r stands for the risk-free discount rate.

Due to the fixed cost, the model violates the requirement that the cost function should be continuously differentiable in the control in order for the Impulse Control Maximum Principle to be applicable. However, performing our transformation of Section 2.2.3 ensures that the Impulse Control Maximum Principle can still be applied. Moreover, the discontinuity in the cost function causes that the sufficient conditions do not hold, i.e. the Impulse Hamiltonian is not concave in (I,v^i) jointly. This implies that we can have multiple solutions satisfying the necessary optimality conditions. To solve this problem, Luhmer (1986) describes an algorithm that finds all these candidate solutions. Typically, this produces a tree structure in which the jumps of all candidate solutions are presented (cf. Section 2.3.6). Given that an optimal solution exists, it is that candidate solution with the highest objective value.

2.3.4 Optimal Dynamic Mix of Manual and Automatic Output (*Case B*)

Gaimon(1985; 1986a) determines the optimal times of impulse acquisition of automation and the change for manual output. The objective is to minimize cost associated with deviation from a goal level of output. The purchase of automation is used to directly substitute for output resulting from manually operated equipment. Since automation is acquired at discrete times in the planning period the author solves the model using the impulse control maximum principle. The following model is given:

$$\begin{aligned} J(T) = \min_{h,s,v,\tau,N} \quad & \int_0^T \Big(w[p(t)+q(t)-g(t)]^2 + c_1(t)h^2(t) \\ & + c_2(t)s^2(t) + f_1(t)p(t) + f_2(t)q(t) \Big) e^{-rt}dt, \\ & + \sum_{i=1}^{N} c_3(\tau_i)v^i e^{-r\tau_i} - \beta[p(T)+q(T)]e^{-rT}, \end{aligned} \tag{2.22}$$

subject to

$$\begin{aligned} &\dot{p}(t) = -d(t) + h(t) - s(t), \quad \text{for } t \notin \{\tau_1, \ldots, \tau_N\}, \\ &q(\tau_i^+) - q(\tau_i^-) = \mu v^i, \qquad \text{for } i \in \{1, \ldots, N\}, \\ &h(t) \in [0, H(t)], \quad s(t) \in [0, S(t)], \quad p(0) = p_0, \\ &q(0^-) = q_0, \qquad v^i \in \{0,1\}, \quad 0 \le \tau_1 < \tau_2 > \ldots, < \tau_N \le T, \end{aligned}$$

where N is the number of times automation equipment is acquired. $c_3(\tau_i)v^i$, $i = 0, ..., N$, the cost of acquiring the ith automation at time τ_i, where v^i denotes the ith technology purchase. The level of automation output and manual output are given by $q(t)$ and $p(t)$ respectively. The cost of producing output manually at time t is given by $f_1(t)$ and the cost of producing output automatically at time t is given by $f_2(t)$. The cost of increasing and reducing the level of manual output per unit squared at time t is represented by $c_1(t)h^2(t)$ and $c_2(t)s^2(t)$, respectively, where $h(t)$ denotes the level of increase in manual output at time t, with $H(t)$ the available supply of labor and $s(t)$ denotes the level of reduction in manual output at time t, with $S(t)$ the maximum permitted level of reduction at time t. The level of reduction in manual output at time t in units of output is represented by $d(t)$, and $g(t)$ represents the goal level of output at time t also in units of output. Finally, w stands for the weight or cost of the squared deviation between the actual and the goal levels of output, μ the units of increase in output due to purchased automation, r is the discount rate, and β the value of the production per unit of output at the end of the planning period.

The difference with the other impulse control models is that the impulse control variable v^i can takes only two values: 0 or 1. It follows that the term $c_3(\tau_i)v^i$ works as a fixed cost. Hence, analogous to the model in Section 2.3.3, sufficient conditions do not hold, so that in principle multiple solutions can satisfy the necessary optimality conditions. Furthermore some transformation as in Section 2.2.3 is needed to apply the Impulse Control Maximum Principle. This is not mentioned in Gaimon (1985; 1986a). A similar reasoning holds for Gaimon and Thompson (1984).

Gaimon (1986b) determines the optimal times and levels of impulse acquisition of automation and the levels of change for manual output with a similar objective. The main difference is that in Gaimon (1986b) the magnitude of automation output can have different values. So Gaimon (1986b) not only determines the time of acquiring automation but also the size of this acquisition. The model is:

$$\begin{aligned} J(T) = \min_{h,s,f_2,v,\tau,N} \quad & \int_0^T \{w[p(t)+q(t)-g(t)]^2 + c_1(t)h^2(t) \\ & +c_2(t)s^2(t) + f_1(t)p(t) + [F_2(t)+f_2(t)]q(t)\}e^{-rt}dt, \\ & +\sum_{i=1}^{N} c_3(v^i,\tau_i)e^{-r\tau_i} - \beta[p(T)+q(T)]e^{-rT}, \end{aligned} \tag{2.23}$$

subject to

$$\begin{aligned} &\dot{p}(t) = -d(t) + h(t) - s(t), \quad \text{for } t \notin \{\tau_1,\ldots,\tau_N\}, \\ &q(\tau_i^+) - q(\tau_i^-) = v^i, \qquad\quad \text{for } i \in \{1,\ldots,N\}, \\ &f_2(\tau_i^+) = f_2(\tau_i^-)[1-\alpha v^i], \\ &h(t) \in [0,H(t)], \quad s(t) \in [0,S(t)], \quad p(0) = p_0, \quad p(t) \geq 0, \\ &q(0^-) = q_0, \qquad v^i \in [0,A(\tau_i)], \quad 0 \leq \tau_1 < \tau_2 > \ldots, < \tau_N \leq T, \end{aligned}$$

where in addition to the notation also used in model (2.22), $F_2(t)$ is the component of the per unit cost of operating automatic equipment that is unaffected by the acquisition of automation at time t, $f_2(t)$ is the per unit cost of obtaining output automatically at time t, whereas α stands for the effectiveness of a unit acquisition of automation on reducing $f_2(\tau_i)$ at time τ_i $(0 \leq \alpha \leq 1/A(\tau_i))$.

All examples in Gaimon (1986b) have an impulse cost function of the form $c_3(v^i,\tau_i) = C_0 + C_1{v^i}^2$. This again implies that the problem contains a fixed cost, and thus sufficiency conditions do not hold so that multiple solutions can satisfy the necessary optimality conditions.

2.3.5 Firm Behavior under a Concave Adjustment Cost Function (*Case C*)

In Kort (1989) a model is given that analyzes the behavior of a firm under a concave adjustment cost function. Kort (1989) applies impulse control because the concave cost function results in a Hamiltonian that is convex in the control. The following model is studied:

$$C(T) = \max_{v,\tau,N} \int_0^T S(K(t))e^{-rt}dt - \sum_{i=1}^{N} \Big(v^i + A(v^i)\Big)e^{-r\tau_i} + K(T)e^{-rT}, \tag{2.24}$$

subject to

$$\begin{array}{ll} \dot{K}(t) = -aK(t), & \text{for } t \notin \{\tau_1, \ldots, \tau_N\}, \\ K(\tau_i^+) - K(\tau_i^-) = v^i, & \text{for } i \in \{1, \ldots, N\}, \\ K(t) \in \mathbb{R}_+, \quad v^i \in (0, \infty) \quad K(0) = K_0, & 0 \leq \tau_1 < \tau_2 > \ldots, < \tau_N \leq T, \end{array}$$

where v^i stands for the i-th investment impulse, and τ_i is the time of the i-th impulse. The adjustment costs of the i-th investment impulse are given by $A(v^i)$ (with $\frac{\partial A(v)}{\partial v} > 0$ and $\frac{\partial^2 A(v)}{\partial v^2} < 0$), $K(t)$ is the amount of capital goods at time t, and a is a constant depreciation rate. Like Feichtinger and Hartl (1986), Kort (1989) applies the incorrect current value Impulse Control Maximum Principle and designs an algorithm to find all candidate solutions that starts at time T and works backward in time (this is different from Luhmer (1986), whose algorithm starts at time zero). The Hamiltonian and Impulse Hamiltonian are

$$\mathcal{H}am(K, \lambda) = S(K) - \lambda aK,$$

$$\mathcal{IH}am(v, \lambda) = -\big(v + A(v)\big) + \lambda v.$$

Note that the Impulse Hamiltonian does not depend on K so here there is no state-control interaction. However the sufficient conditions do not hold due to the concave adjustment cost function which implies that the Impulse Hamiltonian is not concave in v^i. The continuous case of this problem is also described in Kort (1989, pp. 57–62) and consists of a chattering control solution. Consequently, the impulse control model has a "singular" arc with chattering too. Analogous to the Blaquière (1977a; 1977b) model in section 2.3.1, also here we have to conclude that no optimal solution exists. This was not noted in Kort (1989, pp. 62–70).

2.3.6 Dike Height Optimization (*Case C*)

This section analyzes the problem of the optimal timing of heightening a dike. The cost-benefit-economic decision problem contains two types of cost, namely investment cost and cost due to damage (caused by failure of protection by the dikes). Clearly, there is a trade off between investment cost and damage cost. The model in Chahim et al. (2012) is as follows:

$$\min_{v,\tau,N} \left\{ \int_0^T S(t)e^{-rt}dt + \sum_{i=1}^{N} I(v^i, H(\tau_i^-))e^{-r\tau_i} + e^{-rT}\frac{S(T)}{r} \right\}, \tag{2.25}$$

subject to

$$\begin{array}{ll} \dot{H}(t) = 0, & \text{for } t \notin \{\tau_1, \ldots, \tau_N\}, \\ H(\tau_i^+) - H(\tau_i^-) = v^i, & \text{for } i \in \{1, \ldots, N\}, \\ H(t) \in \mathbb{R}_+, \quad v^i \in [0, \infty) & H(0^-) = 0, \quad 0 \le \tau_1 < \tau_2 > \ldots, < \tau_N \le T, \end{array}$$

where v^i stands for the i-th dike heightening, $H(t)$ is the height in centimeter (cm) of the dike at time t relative to the initial situation, i.e. $H(0) = 0$, τ stands for the time of the dike update (years), and r is the risk-free discount rate. The objective (2.25) consists of two parts. The first part is the total (discounted) expected damage cost, which is given by

$$\int_0^T S(t)e^{-rt}dt + \frac{S(T)e^{-rT}}{r},$$

where $S(t)$ denotes the expected damage at time t, $S(t) = P(t)V(t)$. The flood probability $P(t)$ (1/year) in year t is defined as

$$P(t) = P_0 e^{\alpha\eta t} e^{-\alpha H(t)}, \tag{2.26}$$

where α (1/cm) stands for the parameter in the exponential distribution regarding the flood probability, η (cm/year) is the parameter that indicates the increase of the water level per year, and P_0 denotes the flood probability at $t = 0$. The damage of a flood V(t) (million €) is given by

$$V(t) = V_0 e^{\gamma t} e^{\zeta H(t)}, \tag{2.27}$$

in which γ (per year) is the parameter for economic growth, and ζ (1/cm) stands for the damage increase per cm dike height. V_0 (million €) denotes the loss by flooding at time $t = 0$. The second part of the objective is the total (discounted) investment cost

$$\sum_{i=1}^{N} I(v^i, H(\tau_i^-))e^{-r\tau_i},$$

where N is the number of dike heightenings and $H(\tau^-)$ the height of the dike (in cm) just before the dike update at time τ (left-limit of $H(t)$ at $t = \tau$). The investment cost is given by

$$I(v^i, H(\tau^-)) = \begin{cases} A_0(H(\tau^-) + v^i)^2 + b_0 v^i + c_0 & \text{for } v^i > 0 \\ 0 & \text{for } v^i = 0, \end{cases}$$

for suitably chosen constants A_0, b_0 and c_0. The current value Hamiltonian is

$$Ham(t, H(t)) = -S_0 e^{\beta t} e^{-\theta H(t)},$$

while the Impulse Hamiltonian is given by

$$\begin{aligned} IHam(t, H(\tau^-), v^i, \lambda(\tau)) &= -I(v^i, H(\tau_i^-)) + \lambda(\tau)v^i \\ &= -A_0(H(\tau^-) + v^i)^2 - b_0 v^i - c_0 + \lambda(\tau)v^i. \end{aligned}$$

This problem is modeled as an impulse control problem due to the fixed cost, c_0, involved with each dike heightening. As was the case for Luhmer (1986), due to this fixed cost a discontinuity arises in the cost function. The first implication is that the Impulse Control Maximum Principle cannot be straightforwardly applied (although our transformation in Section 2.2.3 makes up for this), and, second, the sufficiency conditions do not hold (i.e. the Impulse Hamiltonian is not concave in (H, v^i) jointly). Chahim et al. (2012) implement the backward algorithm designed by Kort (1989, pp. 62–70). This algorithm solves the above stated problem (2.25) for different values of $H(T)$. We select that $H(T)$, which corresponds to the solution with the lowest value of the objective function. In Figure 2.4 the tree for the Dutch dike ring area 10 is presented. The tree shows all candidate solutions for (the optimal) $H(T) = 282.57$. Due to the fixed costs, small jumps cannot be optimal which is why one can cut away all the upper branches in Figure 2.4. Formally this can be proved by observing that a solution that contains such a small jump, is dominated by a solution where the small jump is deleted, while instead it is added to the previous jump. This implies that only the optimal solution is left. In Table 3.5 this optimal solution (and corresponding cost) are presented.

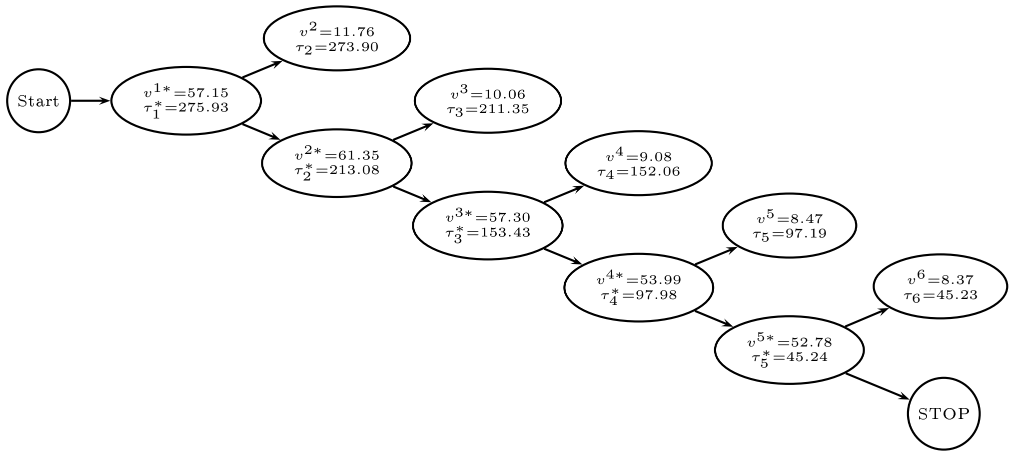


Figure 2.4 – Example Tree, Dike ring area 10, $H(T) = 282.57$.

No.	10	
	τ_i (years)	$v^i(cm)$
Updates	275.93	57.15
	213.08	61.35
	153.43	57.30
	97.98	53.99
	45.24	52.78
$H(T)(cm)$	282.57	
Investment cost (million €)	10.17	
Damage cost (million €)	29.96	
Total cost (million €)	40.13	

Table 2.1 – Impulse control solutions for dike ring area 10 with quadratic investment cost.

2.4 Conclusions and Recommendations

This chapter gives a correct formulation of a necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. In this way we correct Feichtinger and Hartl (1986, Appendix 6) and Kort (1989, pp. 62–70). We review the existing impulse control models in the literature and show that all meaningful problems found in the literature do not satisfy the sufficiency conditions. We observe that these problems either have a concave cost function, contain a fixed cost, or have a control-state interaction, which all lead to non-concavities violating sufficiency. The implication of not satisfying the sufficiency conditions is that multiple solutions can arise and a so called tree-structure of jumps can be identified. We also show that for some problems no optimal solution exists since part of the trajectory consists of staying on the singular arc by applying some kind of impulse chattering. Finally, we provide a transformation, which makes clear why the Impulse Control Maximum Principle can still be applied to problems with a fixed cost despite the fact that this violates the continuous differentiability property of the model.

In this chapter, we classify existing operations research models involving impulse control in four categories. In doing so we observe that *non*-linear deterministic impulse control problems in which a continuous control is present (*case D*) are missing in the literature. Some possibilities for future research arise here. A possibility is to extend Chahim et al. (2012) with continuous dike maintenance.

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CHAPTER 3

An Impulse Control Approach to Dike Height Optimization

Abstract This chapter determines the optimal timing of dike heightenings as well as the corresponding optimal dike heightenings to protect against floods. To derive the optimal policy we design an algorithm based on the Impulse Control Maximum Principle. In this way this chapter presents one of the first real life applications of the Impulse Control Maximum Principle. We show that the proposed Impulse Control approach performs better than dynamic programming with respect to computational time. This is caused by the fact that Impulse Control does not need discretization in time.

3.1 Introduction

In February 1953 the south-western part of the Netherlands was struck by a flood disaster. The flood occurred in the night and resulted into the death of 1,835 people. Almost 200,000 hectares of land were flooded, 3,000 homes and 300 farms destroyed, and 47,000 herd of cattle drowned. In total there were 67 dike breaks. It was the biggest flood in the Netherlands for 300 years. Soon after this flood the Dutch government installed the Delta Committee with the main objective to prevent the occurrence of such events in the future, taking into account that 40% of the Netherlands is below sea level. The Delta Committee asked Van Dantzig (1956) to solve the economic cost-benefit decision model concerning the dike height problem. Because of sea-level rise and economic growth at some specific moments in time the height of the dike must be raised.

In 1995 again a critical situation occurred, where the water level of the major rivers Rhine and Meuse increased so much that 200,000 people where forced to evacuate.

After all, there was no serious flood and people could safely return to their homes. Protection against flooding is becoming an important issue all over the world. There are many deltas that need protection against floods. In Adikari and Yoshitania (2009) it is shown that the total number of natural disasters are continuously increasing in most regions of the world. They state that: *"Among all natural [...] disasters, water-related disasters are undoubtedly the most recurrent and pose major impediments to the achievement and sustainable socio-economic development."*

In Table 3.1 an overview of all recorded natural water-related disasters between 1900-2006 is presented. Between 1900 en 2006 floods accounted for more than 29.8% of

1900-2006	Number of disasters		Number killed ($\times 10^3$)		Total affected ($\times 10^6$)		Real damage US$ ($\times 10^9$)	
Flood	3,050	(42,34%)	6,899	(37,35%)	3,028	(50,18%)	343	(36,07%)
Windstorm	2,758	(38,28%)	10,009	(54,19%)	753	(12,48%)	536	(56,36%)
Drought	836	(11,60%)	1,209	(6,55%)	2,240	(37,12%)	61	(6,41%)
Slides	508	(7,05%)	56	(0,30%)	10	(0,17%)	3	(0,32%)
Wave/Surge	52	(0,72%)	296	(1,60%)	3	(0.05%)	8	(0,84%)
Total	7,204	(100%)	18,469	(100%)	6,034	(100%)	951	(100%)

Table 3.1 – Statistics of recorded natural water-related disasters globally between 1900 and 2006.[1]

the total number of natural disasters (including non-water related, like epidemics, earthquakes etc; see Adikari and Yoshitania (2009)). Of all casualties caused by natural disasters, 18,5% was due to flooding. Moreover more than 48% of the total number of people affected by natural disasters was flood related. In Table 3.2 the number of fatalities due to floods for different parts of the world between 1986 and 2006 are presented. These statistics show that not only the Netherlands, but many regions in the world have to deal with water-related disasters, such as floods. In 2007 the Delta Committee 2 was installed in the Netherlands. The objective of this committee was to advise the Dutch government concerning the consequences of the water level rise for the Dutch coast and the large river deltas. The Delta Committee 2 warned that the sea level could increase more than what was expected in the past. In particular, we should take into account a rise in sea water level between 0.65 meter (m) and 1.30 m around 2100 and a rise between 2 m and 4 m around 2200. In 2009 the Dutch government commissioned a project to develop a cost-benefit analysis and design a method to solve the resulting optimization model in order to set new safety

[1]EM-DAT, The International Disaster Database of the Centre for Research on the Epidemiology of Disasters (CRED).

1986-2006	Number of fatalities	
Asia	117,325	(64.4%)
Africa	14,573	(8.1%)
America	47,782	(26.2%)
Europe	2,120	(1.2%)
Oceania	218	(0.1%)
Total	182,118	(100%)

Table 3.2 – The reported number of fatalities due to floods between 1986 and 2006 per continent.[1]

standards. Results of this project can be found in Hertog and Roos (2009) and Eijgenraam et al. (2011).

This chapter presents an Impulse Control approach as an alternative method to the dynamic programming approach used in Eijgenraam et al. (2011) to solve the dike height problem. Brekelmans et al. (2012) develop a mixed integer nonlinear program (MINLP), but for homogeneous[2] dikes the best approach turns out to be dynamic programming. Therefore we choose to compare it with the Impulse Control approach. To develop the optimal policy we design an algorithm based on the Impulse Control Maximum Principle. We show that the proposed Impulse Control approach performs better than dynamic programming in computation time. This is caused by the fact that Impulse Control does not need discretization in time. Furthermore, this chapter presents one of the first real life applications of the Impulse Control Maximum Principle. In the literature there are not many problems solved using the Impulse Control Maximum Principle. Luhmer (1986) and Kort (1989) design an algorithm to apply the Impulse Control Maximum Principle to theoretically solve (economic) problems. We consider a framework where the number of jumps is not restricted. This distinguishes our approach from, e.g., Liu et al. (1998), Augustin (2002, pp. 71–81) and Wu and Teo (2006), where the number of jumps is fixed (i.e. is taken as given).

The economic cost-benefit problem raised by the flood prevention is formulated by Van Dantzig (1956) as:*"Taking into account, the cost of dikebuilding, the material losses when a dike-break occurs, and the frequency distribution of different sea levels, determine the optimal height of the dikes"*. He assumes that both the economic

[2]A homogeneous dike or dike ring consists of one segment.

value protected by the dikes and the probability of a dike breakthrough are constant over time. In his analysis he determines how much to invest in the heightening of a dike, but did not answer the question "when" to heighten this dike. Eijgenraam et al. (2011) adjusted Van Dantzig's analysis with respect to economic growth. Van Dantzig (1956) found that the height of a dike after every heightening should be such that the resulting flood probabilities are the same. Economic growth implies increasing potential damage, so it is optimal to have lower flood probabilities after every dike height increase. This can be achieved by raising the dike height to higher levels. In this chapter all model assumptions are similar to Eijgenraam et al. (2011).

Impulse Control theory is a variant of optimal control theory where discontinuities (i.e. jumps) in the state variable are allowed. In Impulse Control the moments of these jumps as well as the sizes of the jumps are taken as (new) decision variables. In Blaquière (1985) an example is given that deals with optimal maintenance and life time of machines. Here one has to decide when to replace a machine by a new one (impulse control variable), and has to determine the rate of maintenance expenses (ordinary control variable), so that the profit is maximized over the planning period. In Kort (1989) a dynamic model of the firm is designed in which capital stock jumps upward at discrete points in time at which the firm invests. Blaquière (1977a; 1977b; 1979; 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so-called Impulse Control Maximum Principle, that gives necessary and sufficient optimality conditions for solving such problems.

Blaquière's Impulse Control analysis is based on the present value Hamiltonian form. In this chapter we apply the Impulse Control theorem in the current value Hamiltonian framework as derived in Chahim et al. (2012).

This chapter is organized as follows. In Section 3.2 we first build up the Impulse Control model and derive the necessary optimality conditions. In Section 3.3 we describe the algorithm used to solve the model and obtain an upper bound for the final dike height using the necessary optimality conditions. In Section 3.4 we compare the Impulse Control model to the dynamic programming approach used in Eijgenraam et al. (2011) and present numerical results. Finally, in Section 3.5 we conclude.

3.2 Impulse Control Model

A dike or dike ring is an uninterrupted ring of water defences. There are 53 dike ring areas in the Netherlands with a higher safety standard (i.e. lower flood probability) than 1/1,000 per year. Each dike ring protects a certain area against flooding, see Figure 3.1. The model described in this section can be used for each dike ring separately. In the first section we build up the mathematical model and show that

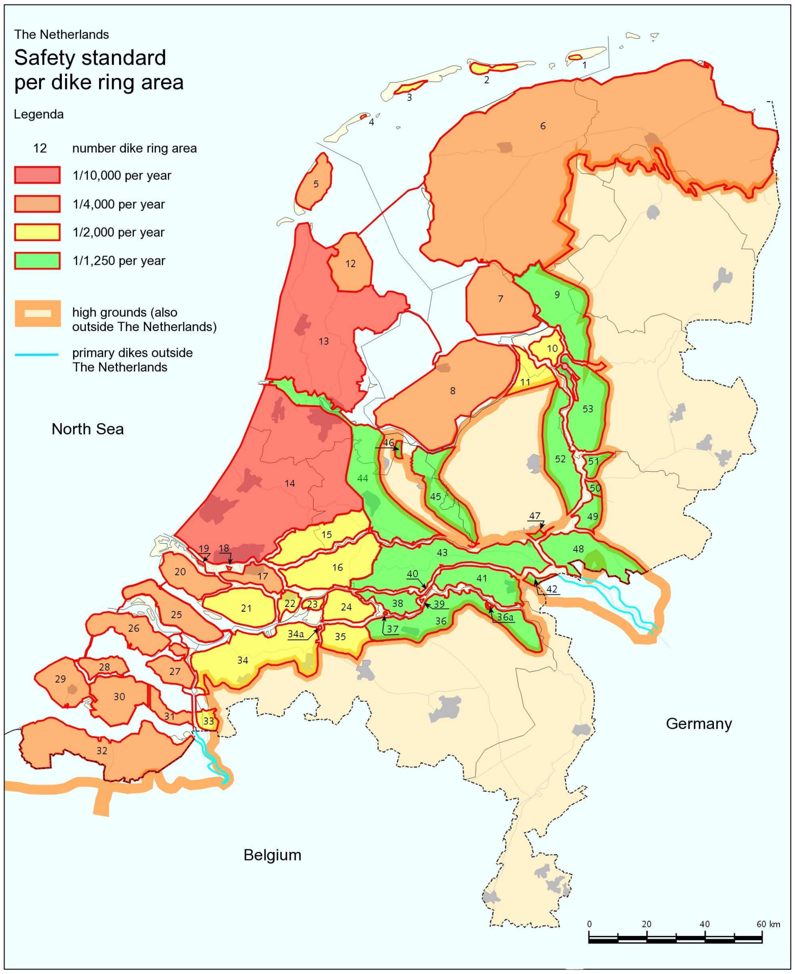


Figure 3.1 – Dike ring areas and safety standards in the Netherlands.

this problem can be described as an Impulse Control problem. In the second section we derive necessary optimality conditions.

3.2.1 The Model

The economic cost-benefit decision problem defined in Eijgenraam (2006) contains two types of cost that we deal with in this problem, namely investment cost and cost due to damage (caused by failure of protection by the dikes). Clearly, there is a trade off between incurring cost due to investing or choosing not to invest and accept the probability that a dike is less protective leading to higher expected damage cost.

The model minimizes the sum of the total expected damage cost and total investment cost. For a thorough discussion of the validity of the underlying model assumptions and parameter values we refer to Eijgenraam et al. (2011).

Let τ (with $0 \leq \tau_1 < \tau_2 < \ldots < \tau_K \leq T$) stand for the time of the dike heightening (years) and $H(t)$ denotes the dike height at time t (years) relative to the initial situation, i.e. $H(0) = 0$ (cm). The investment cost will be denoted as $I(u, H(\tau^-))$, with $H(\tau^-)$ the height of the dike (in cm) just before the dike heightening at time τ (i.e. $H(\tau^-) = \lim_{t\uparrow\tau} H(t)$) and u the amount of the dike heightening. Concerning the investment cost functions, we consider two different specifications. The exponential investment cost function is given by

$$I(u, H(\tau^-)) = \begin{cases} (c_0 + b_0 u)\, e^{a_0 (H(\tau^-)+u)} & \text{for } u > 0 \\ 0 & \text{for } u = 0, \end{cases} \tag{3.1}$$

where a_0, b_0 and c_0 are positive constants. The quadratic investment cost functions is given by

$$I(u, H(\tau^-)) = \begin{cases} a_1 (H(\tau^-) + u)^2 + b_1 u + c_1 & \text{for } u > 0 \\ 0 & \text{for } u = 0, \end{cases} \tag{3.2}$$

for suitably chosen constants a_1, b_1 and c_1. Observe that both the exponential and the quadratic investment cost functions depend on the height of the dike at the moment of heightening. This is contrary to Van Dantzig (1956), who uses a linear cost function that does not depend on the current height of the dike. Our investment cost specifications are in line with the engineering experience that making a dike higher also requires making it wider, implying that an additional dike height increase costs more if the current height is higher (see e.g. Sprong (2008)). Total (discounted) investment cost is then given by

$$\sum_{i=1}^{K} I(u_i, H(\tau_i^-)) e^{-r\tau_i},$$

where r is the discount rate, u_i (cm) denotes the size of the i-th dike heightening, and τ_i is the time of the i-th dike heightening. Following Eijgenraam et al. (2011), we define the flood probability $P(t)$ (1/year) at time t as

$$P(t) = P_0 e^{\alpha\eta t} e^{-\alpha H(t)}, \tag{3.3}$$

where α (1/cm) stands for the parameter in the exponential distribution regarding the flood probability and η (cm/year) is the parameter that represents the increase of the water level per year. The flood probability at time $t = 0$ (i.e. the current flood probability) is denoted by P_0 (1/year), note that $P(0^-) = P_0$. We next describe the value of the damage by a flood, $V(t)$ (million euros):

$$V(t) = V_0 e^{\gamma t} e^{\zeta H(t)}, \tag{3.4}$$

in which γ (per year) is the parameter representing economic growth, and ζ (1/cm) stands for the damage increase per cm dike height. The loss by flooding at time $t = 0$ is denoted by V_0 (million euros). Note that $V(0^-) = V_0$. If $\zeta > 0$ (1/cm), the damage of a flood increases with the height of the dike. The intuition behind this is that when there is a flood, it holds that the higher the dike the longer a high water level will be maintained on the flooded land. This causes higher damage cost. Multiplying the flood probability with the value of the damage by a flood leads to the expected loss due to a flood. From (3.3) and (3.4) it follows that the expected damage at time t equals

$$S(t) = P(t)\, V(t) = S_0 e^{\beta t} e^{-\theta H(t)}, \tag{3.5}$$

with $S_0 = P_0 V_0$, $\beta = \alpha\eta + \gamma$, and $\theta = \alpha - \zeta$.

We consider a finite time horizon $[0, T]$. The total expected damage cost on the time interval $[0, T]$ equals

$$\int_0^T S(t) e^{-rt} dt = \int_0^T S_0 e^{\beta t} e^{-\theta H(t)} e^{-rt} dt,$$

and the expected damage cost after T, the so-called salvage value, is given by

$$S(T) \int_T^\infty e^{-rt} dt = \frac{S(T) e^{-rT}}{r}.$$

Hence, total (discounted) damage cost is given by

$$S_0 \int_0^T e^{\beta t} e^{-\theta H(t)} e^{-rt} dt + \frac{S(T) e^{-rT}}{r}.$$

The aim is to minimize the sum of the investment and expected damage cost:

$$\min \int_0^T S_0 e^{\beta t} e^{-\theta H(t)} e^{-rt} dt + \sum_{i=1}^{K} I\left(u_i, H\left(\tau_i^-\right)\right) e^{-r\tau_i} + e^{-rT} \frac{S(T)}{r},$$

where K is the endogenous number of dike heightening in $[0, T]$.

The height of the dike, $H(t)$, between two dike heightening does not change over time[3]:

$$\dot{H}(t) = 0 \text{ for } t \notin \{\tau_1, \ldots, \tau_K\}.$$

Dike heightenings occur at times $\tau_1, \ldots, \tau_K$. Then we have that

$$H\left(\tau_i^+\right) - H\left(\tau_i^-\right) = u_i \text{ for } i \in \{1, \ldots, K\},$$

where $H(\tau^+)$ denotes the height of the dike (in cm) just after the dike heightening at time τ. The dike heightening problem then becomes

$$\min_{u,\tau,K} \int_0^T S_0 e^{\beta t} e^{-\theta H(t)} e^{-rt} dt + \sum_{i=1}^{K} I\left(u_i, H(\tau_i^-)\right) e^{-r\tau_i} + e^{-rT} \frac{S_0 e^{\beta T} e^{-\theta H(T)}}{r}, \quad (3.6)$$

subject to

$$\begin{aligned}
&H(0^-) = 0, \\
&\dot{H}(t) = 0, && \text{for} \quad t \notin \{\tau_1, \ldots, \tau_K\} \\
&H\left(\tau_i^+\right) - H\left(\tau_i^-\right) = u_i, && \text{for} \quad i \in \{1, \ldots, K\}, \\
&H(t) \in \mathbb{R}_+, \quad u_i \in [0, \infty), \quad 0 \leq \tau_1 < \tau_2 < \ldots, < \tau_N \leq T.
\end{aligned}$$

This is an Impulse Control problem as described in Blaquière (1977a; 1977b; 1979; 1985). Note that this dike heightening model only contains an impulse control variable and not an ordinary control variable. In Blaquière (1979) an example is given of a linear model that contains both an ordinary and an impulse control variable. The example of Blaquière deals with machine maintenance, where the firm has to choose between preventive maintenance (ordinary control) and repair (or upgrade) of the machine (impulse control), see Section 2.3.2.

3.2.2 Necessary Optimality Conditions

In this section we state necessary optimality conditions to solve the Impulse Control dike heightening model given by (3.6). Here we employ the current value Hamiltonian form derived in Chahim et al. (2012). This is done, because the model described in this chapter involves discounting. Other references stating the necessary optimality conditions for impulse control problems are Blaquière (1977a; 1977b; 1979; 1985), Seierstad (1981) and Seierstad and Sydsæter (1987).

To apply the Impulse Control Maximum Principle the functions $S(t)$ and $I(u, H(\tau^-))$

[3] The dike height can decrease slightly due to damage and wear, however these changes are so small that we neglect them in our model.

should be continuously differentiable in H and u_i on $\mathbb{R}_+$. Moreover $S(T)/r$ should be continuously differentiable in $H(T)$ on $\mathbb{R}_+$, and finally that $I(u_i, H(\tau^-))$ is continuous in τ.

The current value Hamiltonian is

$$Ham(t, H) = -S_0 e^{\beta t} e^{-\theta H},$$

and the current value Impulse Hamiltonian is given by

$$IHam(t, H, u, \lambda) = -I(u, H) + \lambda u,$$

in which λ represents the costate variable.

Applying the necessary optimality conditions from Chahim et al. (2012) to our problem yields:

$$\begin{cases} \dot{\lambda}(t) = r\lambda(t) - \theta S_0 e^{\beta t} e^{-\theta H(t)} \quad t \neq \tau_i \ (i = 1, \ldots, K) & (3.7) \\ \lambda(T) = \dfrac{\theta S_0 e^{\beta T} e^{-\theta H(T)}}{r} & (3.8) \\ \lambda\left(\tau_i^+\right) - \lambda\left(\tau_i^-\right) = I_H\left(u_i, H(\tau_i^-)\right) \quad \text{for } i = 1, \ldots, K & (3.9) \\ -I_u\left(u_i, H\left(\tau_i^-\right)\right) + \lambda\left(\tau_i^+\right) = 0 \quad \text{for } i = 1, \ldots, K & (3.10) \\ S_0 e^{\beta \tau_i}\left(e^{-\theta H(\tau_i^-)} - e^{-\theta H(\tau_i^+)}\right) - rI\left(u_i, H(\tau_i^-)\right) & \\ \quad \begin{cases} > 0 \text{ if } \tau_i = 0 \\ = 0 \text{ if } \tau_i \in (0, T) \\ < 0 \text{ if } \tau_i = T \end{cases} \text{for } i = 1, \ldots, K & (3.11) \\ \dfrac{\partial IHam(t, H(t), 0, \lambda(t))}{\partial u} u \leq 0 \quad \text{for } u \geq 0,\ t \neq \tau_i \ (i = 1, \ldots, K), & (3.12) \end{cases}$$

where $\dot{\lambda}(t)$ denotes the time derivative of the costate variable $\lambda(t)$, I_H and I_u denote the partial derivatives of the investment cost function with respect to the state variable $H(t)$ and u, respectively. The state variable $H(t)$ as well as the costate variable $\lambda(t)$ are piecewise-continuous functions in $\mathbb{R}_+$. The domain of the impulse control u is $\mathbb{R}_+$.

When there is no jump (i.e. $t \neq \tau_i$ $(i = 1, \ldots, K)$) equation (3.7) denotes the change of the costate variable and (3.8) gives the transversality condition at the end. Both (3.9) and (3.10) state that at a jump point the marginal cost is equal to the corresponding marginal gains. In equation (3.9) the jump in the costate variable is equal to I_H, which can be interpreted as the marginal investment cost of increasing the dike height just before a dike height jump of size u_i occurs. Equation (3.10) states that the costate variable $\lambda(t)$, which can be interpreted as the reduction in expected damage of an additional centimeter dike increase, equals the investment cost of an additional centimeter of a dike increase, i.e. I_u. When dividing equation (3.11) by the discount rate r, the first term can be interpreted as the decrease of the discounted value of expected future damage due to the increase of the dike at τ_i, while the last term is the investment cost of the dike heightening. So, at the jump point τ_i it must also hold that the total gain of increasing the dike should be equal to the cost of increasing the dike. It follows that optimal behavior requires that the Net Present Value (NPV) of the investment to increase the dike height equals zero. The NPV equals the difference between discounted future gains and current investment cost.

Since $I(u_i, H(\tau^-))$ is not continuously differentiable in u_i (i.e. the derivative at $u_i = 0$ does not exist, due to the fixed cost) one of the conditions for applying the Impulse Control Maximum Principle is violated and we have a problem applying condition (3.12). Chahim et al. (2012) deals with this problem and provides a transformation for the impulse cost function $I(u_i, H(\tau^-))$, which ensures that the application of the Impulse Control Maximum Principle still provides the optimal solution even in the case of a fixed cost. This transformation is based on a continuously differentiable approximation of the impulse cost function (see Section 2.3 of Chahim et al. (2012)). Combining equation (3.12) with the correct approximation implies that $\lim_{\epsilon\downarrow 0} \frac{\partial IHam_\epsilon}{\partial u}(t, H(t), 0, \lambda(t))u = -\infty \cdot u \leq 0$ for every $u \in [0, \infty)$, where $IHam_\epsilon$ is the continuously differentiable approximation of $IHam$. Hence, (3.12) is satisfied, since it holds for all $t \neq \tau_i$ $(i = 1, \ldots, K)$.

3.3 Impulse Control Algorithm for a Dike Ring

In this section we present an algorithm that can be used to solve the problem described in the previous section and explain how we apply the necessary optimality conditions to find all dike heightenings that are candidates for occurrence in our optimal solution. In the algorithm $H(T)$ (i.e. the height of the dike at $t = T$) is a search variable. We show how to obtain an upper bound for the optimal $H(T)$ using the necessary optimality conditions. Finally, we explain how to find the optimal $H(T)$.

3.3.1 Algorithm

In Chahim et al. (2012) it is shown that the Impulse Control sufficient conditions do not hold in all relevant economic problems found in the literature. For our dike height problem the sufficient conditions do not hold due to the fixed cost in the investment cost function, which breaks down the concavity of the Impulse Hamiltonian. Therefore, solutions satisfying the necessary optimality conditions presented in the previous subsection are just candidate optimal solutions. Based on the necessary optimality conditions, we design an algorithm that finds all *candidate* solutions (i.e. a solution that satisfies the necessary optimality conditions). The candidate that minimizes (3.6) is the optimal solution. This algorithm can lead to multiple candidate solutions already described in Luhmer (1986). Contrary to Luhmer, who designs a forward algorithm, we implement a backward algorithm, as described by Kort (1989). This algorithm starts at the horizon date T instead of starting at $t = 0$. We do this since the forward algorithm uses the costate variable $\lambda(0)$ as a search parameter to start the algorithm. In other words, the forward algorithm needs $\lambda(0)$ as input to initialize the algorithm. Contrary to the forward algorithm, the backward algorithm uses the dike height at the end of the planning period, $H(T)$, as the search parameter. Since $\lambda(t)$ is only an auxiliary variable, $\lambda(0)$ is harder to guess than $H(T)$. Moreover, Section 3.3.3 shows that an upper bound for $H(T)$ can be easily derived using the model characteristics. Figure 3.2 shows a flowchart of the algorithm. The next paragraph explains the algorithm in broad terms. In Appendix 3A the algorithm is presented in more detail.

First, we define $\mathcal{X}$ as a set of triples (τ, u, λ) that represents (part of) a solution based on the necessary optimality conditions, $\mathcal{S}$ as the stack (set) of unfinished (partial) solutions, and $\mathcal{C}$ the set of candidate solutions represented by a set of triples. Let t_s denote the time of the earliest update in $\mathcal{X}$ or T if $\mathcal{X}$ is empty. We refer to the flowchart depicted in Figure 3.2 using roman capital numbers. To initialize the algorithm (I) we choose a final dike height $H(T)$ and calculate $\lambda(T)$ via equation (3.8). Then we check whether a dike increase can occur at the horizon date T, and whether it satisfies the necessary optimality conditions (II). If it does not satisfy these conditions, we go via ($IV.i$), where we set $\mathcal{X} = \{(T, 0, \lambda(T))\}$, to ($V$). If the necessary optimality conditions are satisfied, we go via ($IV.ii$), where we set $\mathcal{X} = \{(T, u(T), \lambda(T^-))\}$, to ($V$). In ($V$) we check whether a dike increase can occur at $t = 0$. If a dike heightening at $t = 0$ can occur and satisfies the necessary optimality conditions we save this candidate solution. More precise, in (VII) we add this triple to $\mathcal{X}$, i.e. $\mathcal{X} = \mathcal{X} \cup \{(0, u(0), \lambda(0))\}$ and save this sequence of triples as a candidate solution in (IX), i.e. $\mathcal{C} = \mathcal{C} \cup \{\mathcal{X}\}$. Parallel to this we go to (VI) to find all other candidate solutions (i.e. in (VI) we check whether other candidate solutions can be found, neglecting the jump at $t = 0$). If a dike heightening at $t = 0$ can not occur or does not satisfy the necessary optimality conditions, we go to (VI). In (VI) we solve the necessary optimality conditions to find the set $\mathcal{J}$ of all triples,

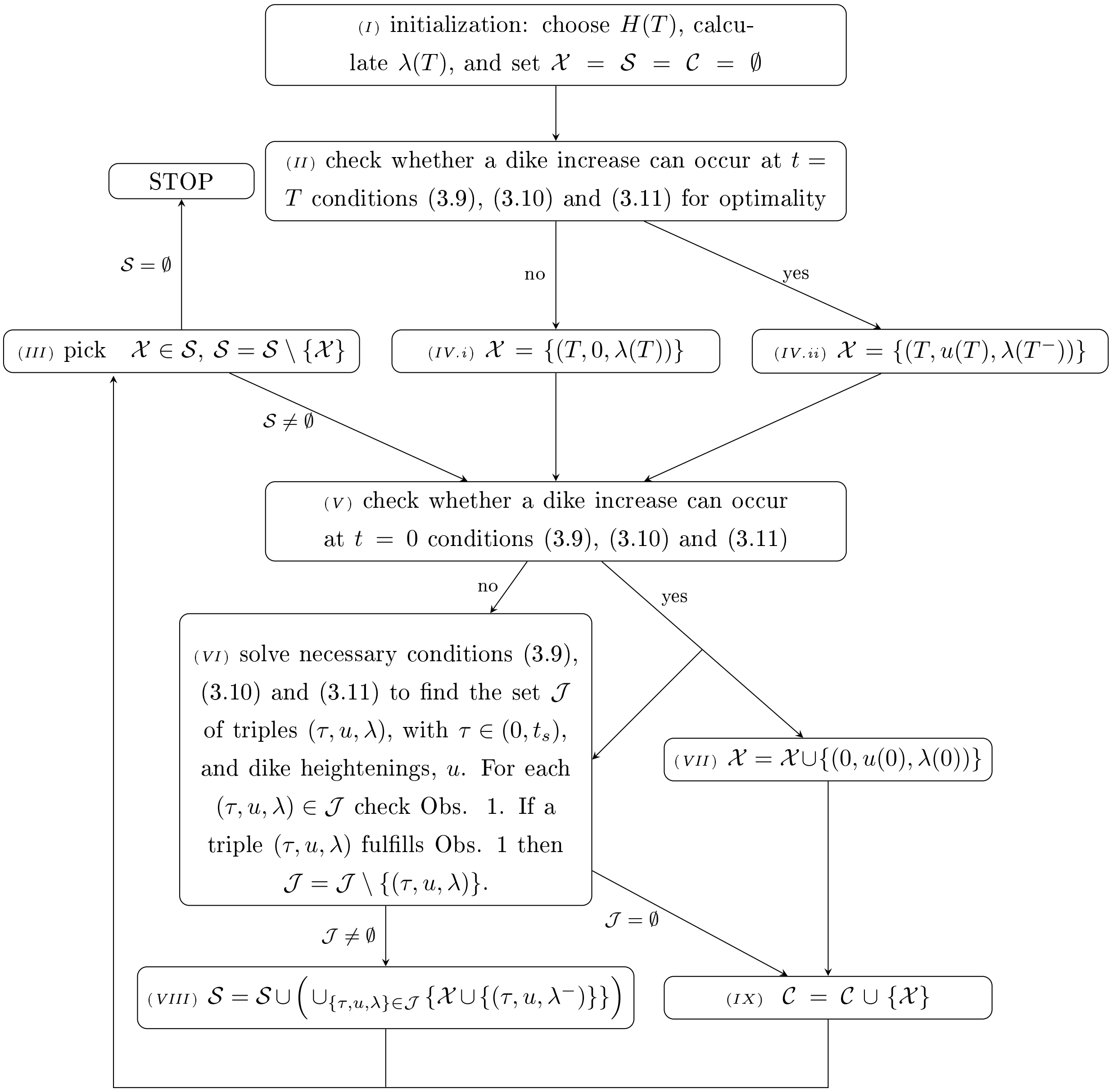


Figure 3.2 – Flowchart of Impulse Control Algorithm for a given $H(T)$.

with $\tau \in (0, t_s)$ and dike heightenings u. If no such triple is found we go to (IX) and save the current sequence $\mathcal{X}$ of triples as a candidate solution. If at least one triple is found, then in $(VIII)$ we add each triple $(\tau, u, \lambda) \in \mathcal{J}$ to the current sequence $\mathcal{X}$, and add the results to the set of unfinished sequences. From $(VIII)$ and (IX) we go to (III) where we pick a sequence $\mathcal{X}$ from the set of open solutions and continue the procedure as shown in Figure 3.2. Finally, if the stack (set) of unfinished (partial) solutions is empty, we stop.

We neglect solutions that are associated with a negative dike heightening, since these are infeasible. Such solutions are discarded and not investigated any further. We also neglect sequences of triples for which the sum of the investment cost for the dike heightening u_j and its predecessor u_{j-1} is larger than the investment cost for increasing the dike with $u_j + u_{j-1}$ at time τ_j. If this is the case, this solution can never be part of the optimal solution since updating with $u_j + u_{j-1}$ at τ_j has lower discounted investment cost and induces more safety (note that $\tau_j < \tau_{j-1}$). This results in the following observation.

Observation 1. If:

(i) $$u_j \leq 0,$$

or

(ii) $$e^{-r\tau_j} I(u_j, H(\tau_j^-)) + e^{-r\tau_{j-1}} I(u_{j-1}, H(\tau_{j-1}^-)) \geq e^{-r\tau_j} I(u_j + u_{j-1}, H(\tau_j^-)),$$

then the corresponding solution can never be optimal.

This approach yields a set of candidates and we select the candidate with the lowest expected cost. Furthermore, we have to check whether $H(0) = 0$. If this is not satisfied, then the initial $H(T)$ is not optimal and we restart the algorithm with a new initial $H(T)$, more on this in Section 3.3.4.

3.3.2 Solving the Necessary Optimality Conditions

In Figure 3.2 it stated in box (VI) that the necessary optimality conditions are used to find all candidate solutions, i.e. all candidate dike heightenings. Equation (3.10) is of the following form

$$y_1 e^{\alpha_1 t} + y_2 e^{\alpha_2 t} + y_3 e^{\alpha_3 t} - I_u = 0, \tag{3.13}$$

where y_1, y_2, y_3, α_1, α_2 and α_3 are constants. Expression (3.11) is of the following form:

$$z_1 e^{\beta_1 t} + z_2 e^{\beta_2 t + \beta_3 u} - rI \begin{cases} > 0 \text{ for } t = 0 \\ = 0 \text{ for } t \in (0, T) \\ < 0 \text{ for } t = T, \end{cases} \tag{3.14}$$

where z_1, z_2, β_1, β_2 and β_3 are constants. If (3.13) depends on u and t this can be rewritten into a function $u(t)$ which can be substituted into (3.14). The resulting non-linear equation has only one unknown t. Solving this leads to all possible jumps points τ, and $u(\tau)$ gives us the corresponding jump size. It can also be the case that (3.13) depends only on t. Then (3.13) can be solved to find all τ. Using (3.14) we find all corresponding jump sizes u. Finally, equation (3.9) gives us the value of the costate variable before the dike update. This results in a set of triples (τ, u, λ).

3.3.3 Finding an Upper Bound for the Optimal Ending Dike Height

Let $H^*(T)$ denote the end dike height (i.e. the height at $t = T$) of the optimal solution to our problem (3.6). An upper bound can be obtained by using the necessary optimality conditions. Investing in a dike is only "profitable" if the marginal cost of the investment is at most equal to the marginal revenue. In the cases of exponential and quadratic cost function the following results can be established.

Proposition 3.3.1.
For exponential cost (see (3.1)):

Let $T > \frac{1}{\beta} \ln \frac{r(b_0+a_0c_0)}{\theta S_0}$*, and let* $\bar{H}_e$ *be defined by the solution of the following equation:*

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_e}}{r} = b_0 e^{a_0 \bar{H}_e} + a_0 c_0 e^{a_0 \bar{H}_e}. \tag{3.15}$$

Furthermore, let

$$\hat{H}_e = \frac{1}{\theta + a_0} \ln \left(\frac{\theta S_0 e^{\beta T}}{r b_0} \right).$$

Then, it holds that $H^*(T) \leq \bar{H}_e \leq \hat{H}_e$*.*

For quadratic cost (see (3.2)):

Let $T > \frac{1}{\beta} \ln \frac{r b_1}{\theta S_0}$*, and let* $\bar{H}_q$ *be defined by the solution of the following equation:*

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_q}}{r} = 2a_1 \bar{H}_q + b_1.$$

Furthermore, let

$$\hat{H}_q = \frac{1}{\theta} \ln \left(\frac{\theta S_0 e^{\beta T}}{r b_1} \right).$$

Then, it holds that $H^*(T) \leq \bar{H}_q \leq \hat{H}_q$.

Proof: An upper bound for $H^*(T)$ is the end height for which the following equation (3.10) holds at time horizon T:

$$\lambda(T^+) \quad = \quad I_u\left(u_i, H^*(T)\right),$$

with

$$\lambda\left(T^+\right) = \lambda\left(T\right) = \frac{\theta S_0 e^{\beta T} e^{-\theta H^*(T)}}{r}.$$

For exponential investment cost this (with no dike heightening at $t = T$) boils down to solving the following equation:

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_e}}{r} = b_0 e^{a_0 \bar{H}_e} + a_0(c_0 + b_0 u) e^{a_0 \bar{H}_e}, \tag{3.16}$$

where $\bar{H}_e$ denotes the upper bound for $H^*(T)$. The left-hand side of (3.16) gives the marginal gain of a dike heightening and is decreasing in $\bar{H}_e$. The right-hand side of (3.16) gives the marginal cost of such a heightening and is increasing in $\bar{H}_e$. We lower the right-hand side of (3.16) by omitting $a_0 b_0 u e^{a_0 \bar{H}_e}$; this shifts the graph to the right and results in a lower marginal cost at $t = T$. Additionally, this gives us equation (3.15). Since $T > \frac{1}{\beta} \ln \frac{r(b_0 + a_0 c_0)}{\theta S_0}$, we have that the left-hand side of (3.15) is larger than the right-hand side of (3.15) at $\bar{H}_e = 0$. Combining the latter with the fact that left-hand side of (3.15) is decreasing in $\bar{H}_e$, that the right-hand side of (3.15) is increasing in $\bar{H}_e$, that

$$\lim_{\bar{H}_e \to \infty} b_0 e^{a_0 \bar{H}_e} + a_0 c_0 e^{a_0 \bar{H}_e} = \infty,$$

and that

$$\lim_{\bar{H}_e \to \infty} \frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_e}}{r} = 0,$$

results in a unique solution $\bar{H}_e$ for equation (3.15). Furthermore, we lower the right-hand side of (3.15) by now omitting $a_0(c_0 + b_0 u) e^{a_0 \bar{H}_e}$, this again shifts the graph of the right-hand side to the right and results in a lower marginal cost at $t = T$. Hence, an upper bound for $\bar{H}_e$ results from

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \hat{H}_e}}{r} = b_0 e^{a_0 \hat{H}_e}, \tag{3.17}$$

where $\hat{H}_e$ denotes the upper bound for $\bar{H}_e$ (i.e. $H^*(T) \leq \bar{H}_e \leq \hat{H}_e$). Solving (3.17) we get that $\hat{H}_e$ is given by:

$$\hat{H}_e = \frac{1}{\theta + a_0} \ln \left(\frac{\theta S_0 e^{\beta T}}{r b_0} \right).$$

The proof for the quadratic cost function is analogous. ■

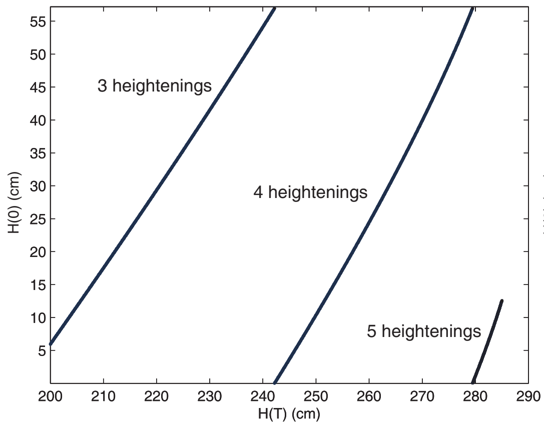
Note that these upper bounds for $H(T)$ can also be used for the dynamic programming approach in Eijgenraam et al. (2011) to decrease the number of states, see Section 3.4.2. Moreover, we have that $\theta S_0 > r(b_0 + a_0 c_0)$ and $\theta S_0 > r b_1$ for all dikes (in the Netherlands).[4] Hence, we have that the condition on T for both cost functions is always satisfied.

3.3.4 Finding the Optimal Ending Dike Height

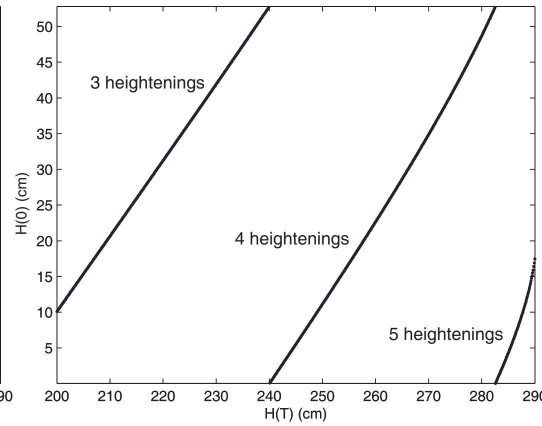
Recall that an ending dike height $H(T)$ is required as an input to the algorithm in Section 3.3. For an arbitrary $H(T)$, the algorithm is not guaranteed to produce a feasible solution to problem (3.6), because the condition on the initial height $H(0) = 0$ might be violated. In that case we always have $H(0) > 0$–since negative heightenings are not allowed–and apparently there does not exist a feasible solution for the chosen $H(T)$ that satisfies all necessary optimality conditions. Thus, we need a procedure to find an ending dike height for which the algorithm returns a feasible solution.

If we find all ending heights for which the algorithm returns feasible solutions, then we know that the optimal solution must be among them, because all solutions, by construction, satisfy all necessary optimality conditions—and there are no other solutions with this property. The dependency on $H(T)$ of any solution produced by the algorithm is piecewise continuous, with discontinuities occurring when the total number of heightenings in $[0, T]$ changes. This is illustrated by Figure 3.3, which shows the residual height $H(0)$ corresponding to the candidate solution that results from the selected ending height $H(T)$. At each discontinuity point the total number of heightenings changes as indicated in the figure. Hence, a bisection method on $H(0)$ could be used to search for an ending height that produces a feasible solution, i.e., $H(0) = 0$. For now, we propose the simpler approach of discretization of $H(T)$ as is also necessary for the dynamic programming approach to the problem (see Eijgenraam et al. (2011)) An upper bound for the discretization of $H(T)$ is readily provided by $\bar{H}$ (see Section 3.3.3) and a suitable lower bound is the current dike height plus the

[4] The data is provided by Rijkswaterstaat, part of the Dutch ministry of Infrastructure and Environment.



(a) Exponential investment cost.



(b) Quadratic investment cost.

Figure 3.3 – Plot of the residual height (i.e. $H(0)$) vs. $H(T)$ for dike 10.

(future) sea-level rise. The set of solutions produced by the algorithm applied to a discretization of $H(T)$ in this range is unlikely to contain exact feasible solutions. To cope with the infeasibility of the solutions, we transform each solution to a feasible solution by adding the residual height $H(0)$ to the solution's first heightening. In that way, both the investment cost of the first heightening and the expected damage from $t = 0$ until the first heightening increase, which makes that there is some difference between the objective values of the original and the transformed solution. Note that if the residual height $H(0)$ is small–and for any reasonably fine grid, solutions with $H(0)$ close to zero should be found–then this difference will be small as well. Of all transformed solutions obtained in this way, we pick the one with the smallest objective value.

3.4 Comparing Impulse Control to Dynamic Programming

This section consists of two parts. First, we compare the numerical results obtained using the Impulse Control approach to the results found in Eijgenraam et al. (2011) using dynamic programming. Second, we derive the computation time of both methods.

3.4.1 Numerical Results for Five Dike Rings

In this section we apply the algorithm described in Section 3.3. The data used in this section are taken from Hertog and Roos (2009) and are presented in Table 3.3.

The data are made available by Rijkswaterstaat/Deltares (i.e. a bureau concerned with practical execution of the public works and water management part of the Dutch Ministry of Infrastructure and the Environment) and were generated by water experts. It is clear that the choice of T will influence the solution. If we choose T too small then this can affect the solution in the beginning of the planning period. We choose T such that the solution in the beginning of the planning period remains stable when T increases. As in Eijgenraam et al. (2011) we set $T = 300$. Taking $T = 600$ gives similar results for the beginning of the planning period compared to $T = 300$. This is caused by the fact that the discount factor ($e^{-0.04*300} \approx 0.00000614$) is small for large values of t. Hence, the effect of the salvage value is very small when $T = 300$. In Tables 3.4 and 3.5 the solutions obtained by using the algorithm described in Section 3.3 for both exponential and quadratic investment cost can be found.

Dike No.	10	11	15	16	22
a_0	0.0014	0	0.0098	0.01	0.0066
b_0	0.6258	1.7068	1.1268	2.1304	0.9325
c_0	16.6939	42.62	125.6422	324.6287	154.4388
a_1	0.0004	0	0.027	0.102	0.0154
b_1	0.7637	1.7168	3.779	3.1956	2.199
c_1	12.603	42.3003	67.699	319.25	141.01
V_0	1564.9	1700.1	11810.4	22656.5	9641.1
r	0.04	0.04	0.04	0.04	0.04
P_0	1/2270	1/855	1/729	1/906	1/1802
H_0	0	0	0	0	0
α	0.033027	0.032	0.0502	0.0574	0.07
η	0.32	0.32	0.76	0.76	0.62
γ	0.02	0.02	0.02	0.02	0.02
ζ	0.003774	0.003469	0.003764	0.002032	0.002893

Table 3.3 – Parameter values for dikes 10, 11, 15, 16 and 22.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	272.8 : 52.18	275.9 : 54.56	259.2 : 57.33	271.6 : 47.89	261.6 : 50.97
	217.0 : 56.43	218.9 : 61.71	206.2 : 54.16	219.2 : 51.69	199.9 : 53.37
	160.1 : 56.90	160.2 : 62.35	154.3 : 53.47	165.3 : 52.41	137.6 : 53.65
	103.0 : 56.95	101.3 : 62.42	103.7 : 53.32	111.5 : 52.55	75.2 : 53.68
	45.9 : 56.96	42.4 : 62.42	51.2 : 53.29	57.5 : 52.57	12.7 : 53.71
			0 : 55.82	3.5 : 52.58	
$H(T)$	279.41	303.47	327.39	309.69	265.37
$\bar{H}_e$	290.93	311.48	347.14	320.48	278.75
$\hat{H}_e$	292.12	311.48	360.28	334.65	288.77
Investment cost	10.16	30.18	414.59	797.75	198.42
Damage cost	29.87	80.05	130.55	291.84	110.82
Total cost	40.03	110.23	545.14	1089.59	309.24

Table 3.4 – Impulse Control solutions for dikes 10, 11, 15, 16 and 22, with exponential cost function.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	275.9 : 57.15	274.6 : 55.09	282.0 : 62.62	245.3 : 76.90	262.1 : 56.36
	213.0 : 61.35	217.8 : 61.39	214.1 : 77.43	176.7 : 69.35	194.5 : 58.53
	153.4 : 57.30	159.4 : 61.97	149.7 : 69.92	113.8 : 61.03	130.5 : 54.13
	98.0 : 53.99	100.9 : 62.03	92.3 : 59.86	56.9 : 52.51	70.7 : 50.15
	45.2 : 52.78	42.4 : 62.05	42.6 : 49.39	3.2 : 48.25	12.7 : 49.74
			0 : 46.44		
$H(T)$	282.57	302.53	365.66	308.04	268.91
$\bar{H}_q$	290.22	311.28	370.28	331.79	283.82
$\hat{H}_q$	299.30	311.28	410.25	387.76	304.39
Investment cost	10.17	30.16	421.30	822.41	201.35
Damage cost	29.96	80.06	160.91	334.72	115.74
Total cost	40.13	110.23	582.21	1157.13	317.09

Table 3.5 – Impulse Control solutions for dikes 10, 11, 15, 16, 22, with quadratic cost function.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	274 : 51.84	272 :42.24	262 : 54.72	274 : 45.60	254 : 52.08
	219 : 55.68	218 :59.52	209 : 54.72	223 : 50.16	194 : 52.08
	162 : 57.60	160 :61.44	156 : 54.72	171 : 50.16	133 : 52.08
	104 : 57.60	101 :63.36	103 : 54.72	116 : 54.72	73 : 52.08
	46 : 57.60	43 :61.44	50 : 54.72	60 : 54.72	12 : 52.08
			0 : 54.72	4 : 54.72	
$H(T)$	280.32	288.00	328.32	310.08	260.4
Investment cost	10.16	29.33	413.39	796.31	202.09
Damage cost	29.87	80.90	131.95	294.13	107.33
Total cost	40.04	110.24	545.34	1090.44	309.41

Table 3.6 – Dynamic programming solutions for dikes 10, 11 15, 16, 22, with exponential cost function.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	277 : 55.68	272 : 42.24	280 : 63.84	274 : 45.60	265 : 55.80
	214 : 61.44	218 : 59.52	212 : 77.52	223 : 50.16	197 : 59.52
	155 : 57.60	160 : 61.44	149 : 68.40	171 : 50.16	131 : 55.80
	99 : 53.76	101 : 63.36	92 : 59.28	116 : 54.72	69 : 52.08
	46 : 53.76	43 : 61.44	42 : 50.16	60 : 54.72	12 : 48.36
			0 : 45.60	4 : 54.72	
$H(T)$	282.24	288.00	364.80	310.08	271.56
Investment cost	9.97	29.33	418.94	840.70	208.15
Damage cost	30.17	80.90	163.35	317.51	112.09
Total cost	40.14	110.24	582.28	1158.21	317.24

Table 3.7 – Dynamic programming solutions for dikes 10, 11, 15, 16, 22, with quadratic cost function.

After comparing the results presented in Table 3.4 and 3.5 with the dynamic programming results taken from Eijgenraam et al. (2011) presented in Table 5.5 and 3.7, we can make the following observations:

- The (total) cost using the Impulse Control approach is always lower. The reason for this (minor) difference is due to the discretization of the problem in time and dike height in the dynamic programming approach.

- Comparing the results between the exponential and quadratic investment cost functions for the Impulse Control approach given in Table 3.4 and 3.5, respectively, no significant difference can be found. The first dike heightening for Impulse control using a quadratic cost function takes place slightly earlier comparing it with the exponential cost function. However, the corresponding amount of this first dike heightening is lower. This difference is also observed for the dynamic programming approach.

- Dike 15 needs to be heightened immediately (i.e. at $\tau_1 = 0$). This result is found for both the exponential and the quadratic cost function, and for both approaches.

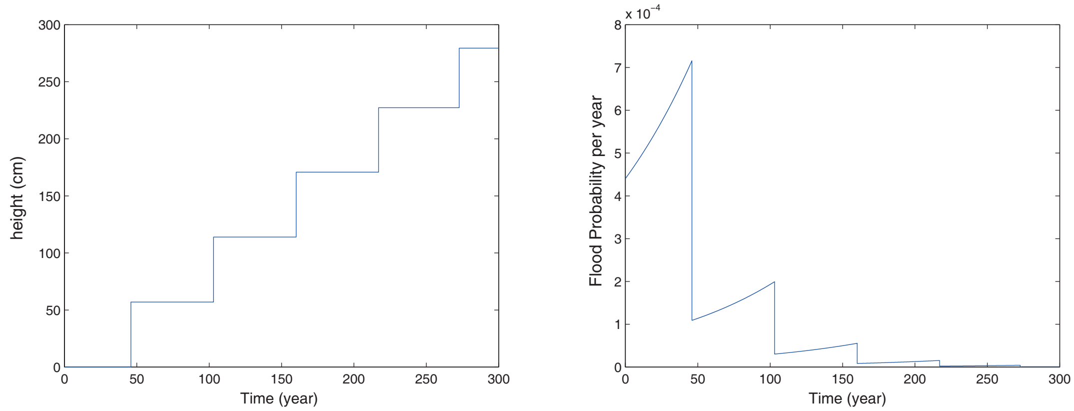
- The Impulse Control approach results in a significantly higher $H(T)$ for dike 11 compared to the dynamic programming approach. This is observed for both cost functions.

- For exponential investment cost the upper bound $\bar{H}_e$ is very close to the optimal $H(T)$ found for all five dikes. Comparing the upper bound for quadratic cost, $\bar{H}_q$, with $\bar{H}_e$ we observe that $\bar{H}_q$ is higher than $\bar{H}_e$ for dikes 15, 16 and 20. The values are comparable for dike rings 10 and 11.

- When the first dike heightening is far from time zero, $\bar{H}_e$ and $\hat{H}_e$ are closer to each other (same holds for $\bar{H}_q$ and $\hat{H}_q$). For dike ring 11 we have that $a_1 = a_0 = 0$ and hence $\bar{H}_e = \hat{H}_e$ and $\bar{H}_q = \hat{H}_q$.

In Figures 3.4 and 3.5 the optimal dike height and the corresponding flood probability of dike 10 are presented for the exponential and quadratic investment cost, respectively. It is striking to see that the upper bound(s) are very close to the optimal dike height at time T. Finally, in Figures 3.4 and 3.5 one can observe that at the time moments where a dike heightening occurs the flood probability drops instantaneously.

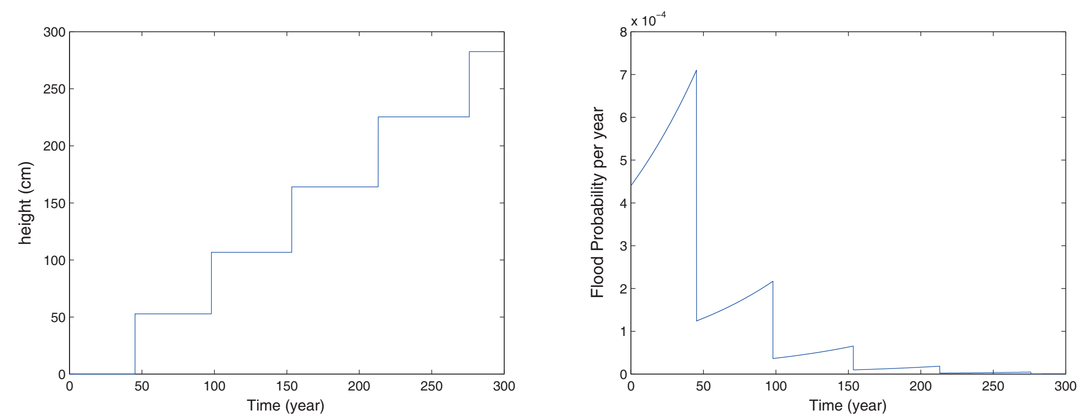
We also observe that after each dike heightening at most three candidate dike heightenings were found by the algorithm (stage VI). In case of three candidates we always found that two out of the three candidates could not be optimal, since one was always negative (Observation 1, (i))) and for the other one it holds that combining this heightening with its predecessor was an improvement (see Observation 1, (ii)).



(a) Dike height over time.

(b) Flood probability per year corresponding to the optimal dike height.

Figure 3.4 – Optimal dike height of dike 10 and the corresponding flood probability using exponential investment cost.



(a) Dike height over time.

(b) Flood probability per year corresponding to the optimal dike height.

Figure 3.5 – Optimal dike height of dike 10 and the corresponding flood probability using quadratic investment cost.

3.4.2 Computation Time

In Section 4 of Eijgenraam et al. (2011) a dynamic programming approach is described that corresponds to the above described Impulse Control model. A drawback of this approach is that the finite time horizon $[0, T]$ as well as the height of the dike $H(t)$ are discretized. This directly has an effect on the accuracy of the method. This can partly be resolved by taking a "finer" discretization. However, this will affect the computation time of the problem. The discretization chosen in Eijgenraam et al. (2011) seemed to be fine enough for the dike heightening problem.

Dynamic programming (DP)

The number of computations that have to be made in the DP approach depends on the number of stages and states (for each stage the value of each state should be calculated). The problem is discretized in both time and dike height. Let us call M the number of states per stage and T the number of stages. Cost are related to the transition from one state to another. The DP problem can be presented in a graph where the vertices in the graph are the states, and the arcs of the graph represent the transition. The aim of DP is to find the shortest path in the graph. In the DP approach used to solve the dike heightening problem in Eijgenraam et al. (2011), the stages defined as the years $t = -1, 0, 1, 2, ...T$, in which $t = -1$ is the time just before $t = 0$. The state at stage t is defined as $H(t)$. For the initial state at stage $t = -1$ it holds that $H(-1) = 0$. Also we know that a transition can only occur from state $H(t)$ in stage t to state $H(t+1)$ in stage $t+1$ such that $H(t+1) \geq H(t)$. In Eijgenraam et al. (2011) the discounted investment and damage cost in the period $[t, t+1]$, for $t = 0, 1, \ldots, T-1$ are given by

$$c_t(H(t), H(t+1)) = \int_t^{t+1} S(t)e^{-rt}dt + I(H(t+1) - H(t), H(t))e^{-r(t+1)},$$

and for $t = -1$, by

$$c_{-1}(H(-1), H(0)) = I(H(0) - H(-1), H(0)) = I(0, 0).$$

The recursive relation for the DP approach is

$$f_t(H(T)) = \min_{H(t)<H(t+1)\in\mathcal{H}_{t+1}} \{f_{t+1}(H(t+1)+c_t(H(t), H(t+1))\}, \quad t < T, \quad H(T) \in \mathcal{H}_t,$$

where $\mathcal{H}_t$ denotes the set of all feasible dike heights at time t. Starting in state $H(T)$, $f_t(H(T))$ denotes the minimal cost to cover the years $t, t+1, \ldots, T, T+1, \ldots, \infty$. The costs after $t = T$ are given by

$$f_T(H(T)) = \frac{S(T)e^{-rT}}{r}.$$

It is easy to see that this DP approach is of order $O(\alpha_{DP} M^2 T)$, where α_{DP} denotes the basic operations needed to calculate the transitions cost from one state to another.

Impulse Control (IC)
Let J be the number of dike heightenings found. To make an easy comparison with DP we run the algorithm for the same candidate final dike heights, i.e. we take the states used in the DP approach as input determining the optimal final dike height. In the dynamic programming approach for each stage a certain number of discretized states are defined. Clearly, for the impulse control approach this is not necessary. Let us call the number of basic operations needed to solve the necessary optimality conditions (see Section 3.3.2) to find all candidate dike heightenings α_{IC}. Then it is easy to see that this problem is of order $O(\alpha_{IC}JM)$. In the previous section we have seen that the number of dike heightenings (5 or 6) in the dike heightening problem never exceeds the number of states ($M = 300$) used in the DP approach and α_{DP} and α_{IC} are comparable. Hence, we can conclude that IC needs less computation time than DP.

3.5 Conclusions and Recommendations

In this chapter we present the first real life application of the Impulse Control Maximum Principle. In doing so, we propose an alternative for the dynamic programming approach used in Eijgenraam et al. (2011). We show that, compared to the dynamic programming approach, the Impulse Control approach has lower computation time. This can be explained since the Impulse Control approach does not need discretization in time and only discretization for the dike height at the end of the time horizon (final stage), unlike dynamic programming where discretization is needed for time and for the heights (states) for each stage. Comparing the total cost for the dike updating scheme for the five dikes presented in this chapter with the total cost using the dynamic programming approach, we observe that the total cost for the Impulse Control approach is always lower. However, the differences are very small. Further, we identify upper bounds for the final dike height, by using the necessary optimality conditions at the end of the planning period. It is striking to see that both proposed upper bounds are very close to the optimal dike height at the end of the planning period. The way we derive these upper bounds can be used in general, so that these upper bounds can also be implemented in the dynamic programming approach. We show that the Impulse Control approach works well for both exponential and quadratic investment cost.

A possible extension of this chapter would be adding some preventive dike maintenance. It would be interesting to analyze the interaction between preventive dike maintenance and the impulse dike heightening. This extension will quadratically increase the number of states for the dynamic programming approach and hence take more time to compute. Another possible extension is applying Impulse Control to nonhomogeneous dikes (i.e. dikes or dike rings that consist of multiple segments) for which the dynamic programming approach is not useful since it suffers from the well-known combinatorial explosion. Also other maintenance problems can be considered.

Appendix 3A: Backward Algorithm for Impulse Control

In this section the algorithm described in Section 3.3 is presented in more detail. Before we start we define $\mathcal{X}$ as a sequence of triples (τ, u, λ), $\mathcal{S}$ the stack (set) of open solutions, and $\mathcal{C}$ the set of candidate solutions. We need one more variable t_s defined as

$$t_s = \begin{cases} T & \text{if} \quad \mathcal{X} = \emptyset, \\ \min_{(\tau,u,\lambda)\in\mathcal{X}} \tau & \text{if} \quad \mathcal{X} \neq \emptyset. \end{cases}$$

Step I: Initialization:

Choose $H(T)$.
Determine the value of the co-state variable:

$$\lambda\left(T\right) = \frac{\theta S_0 e^{\beta T} e^{-\theta H(T)}}{r}.$$

Step II: Check whether a dike height increase can occur at time $t = T$ and whether it is optimal. Derive $H\left(T^-\right)$ and $u\left(T\right)$ from

$$H\left(T^+\right) - H\left(T^-\right) = u\left(T\right),$$

and

$$-I_u\left(u(T), H\left(T^-\right)\right) + \lambda\left(T^+\right) = 0.$$

The dike height increase is optimal at time T if

$$-S_0 e^{\beta T} e^{-\theta H\left(T^+\right)} + S_0 e^{\beta T} e^{-\theta H\left(T^-\right)} - rI\left(u\left(T\right), H\left(T^-\right)\right) < 0.$$

If so, go to step $IV.ii$. Otherwise, define

$$\begin{aligned} H\left(T^+\right) &= H\left(T\right), \\ \lambda\left(T^+\right) &= \lambda\left(T\right), \end{aligned}$$

and go to step $IV.i$.

Step III: If $\mathcal{S} = \emptyset$ STOP. Else pick $\mathcal{X} \in \mathcal{S}$, set $\mathcal{S} = \mathcal{S} \setminus \{\mathcal{X}\}$ and go to step V.

Step $IV.i$: Set $\mathcal{X} = \{(T, 0, \lambda(T))\}$.

Step $IV.ii$: Set $\mathcal{X} = \{(T, u(T), \lambda(T^-))\}$.

Step V: Check whether a dike height increase can occur at time 0 and whether it is optimal.
Solve (3.10) to find $u(0)$. The dike height increase is optimal if

$$-S_0 e^{\beta T} e^{-\theta H\left(0^+\right)} + S_0 e^{\beta T} e^{-\theta H\left(0^-\right)} - rI\left(u\left(0\right), H\left(0^-\right)\right) > 0.$$

If so, go to VI and to VII. If not, go to step VI.

Step VI: Find all $\tau \in (0, t_s)$ such that

$$\lambda(t^+) = e^{-r(t_s - t)}\lambda(t_s) + \int_t^{t_s} e^{-r(s-t)}\theta S_0 e^{\beta s} e^{-\theta(H(t_s))} ds. \tag{3.18}$$

At the point in time where a dike increase can occur, equations (3.9), (3.10) and (3.11) hold.
Combining equation (3.10) and (3.18) gives a condition that holds at the jump point

$$e^{-r(t_s - t)}\lambda(t_s) + \int_t^{t_s} e^{-r(s-t)}\theta S_0 e^{\beta s} e^{-\theta(H(t_s))} ds - I_u(u, H(t_s)) = 0. \tag{3.19}$$

Solving equation (3.19) results either in an explicit function $u(t)$ for the dike heightening or gives all τ for which (3.19) holds. When $u(t)$ can explicitly be identified, go to step $IV.i$, else go to step $VI.ii$.

Step $VI.i$: Substituting $u(t)$ in equation (3.11) yields

$$-S_0 e^{\beta t} e^{-\theta H(t_s)} + S_0 e^{\beta t} e^{-\theta(H(t_s) - u(t))} - rI(u(t), H(t_s)) = 0, \tag{3.20}$$

which is an equation that only depends on t and holds for each jump point $\tau \in (0, t_s)$.

If equation (3.20) is solvable, it gives us all potential jump points τ. Using this, we get all dike heightenings u using $u(t)$ (from equation (3.19)). This gives a set

$\mathcal{J}$ of triples (τ, u, λ). For each triple $(\tau, u, \lambda) \in \mathcal{J}$ check Observation 1 conditions (i) and (ii). If a triple (τ, u, λ) satisfies condition (i) or (ii) of Observation 1 then $\mathcal{J} = \mathcal{J} \setminus \{(\tau, u, \lambda)\}$. If $\mathcal{J} \neq \emptyset$, go to *VIII*, else go to step *IX*.

Step *VI.ii*: For each τ found in step V solve

$$-S_0 e^{\beta\tau} e^{-\theta H(t_s)} + S_0 e^{\beta\tau} e^{-\theta(H(t_s)-u(\tau))} - rI(u(\tau), H(t_s)) = 0, \tag{3.21}$$

to find the corresponding u. This gives a set $\mathcal{J}$ of triples (τ, u, λ). For each triple $(\tau, u, \lambda) \in \mathcal{J}$ check Observation 1. If a triple (τ, u, λ) fulfills Observation 1 then $\mathcal{J} = \mathcal{J} \setminus \{(\tau, u, \lambda)\}$. If $\mathcal{J} \neq \emptyset$, go to *VIII*, else go to step *IX*.

Step*VII*: Save $\mathcal{X} = \mathcal{X} \cup \{(0, u(0), \lambda(0))\}$ and go to step *IX*.

Step *VIII*: Add each triple $(\tau, u, \lambda) \in \mathcal{J}$ to the current sequence $\mathcal{X}$ and add the result to the stack (set) of unfinished (partial) solutions, i.e.:

$$\mathcal{S} = \mathcal{S} \cup \Big(\cup_{(\tau,u,\lambda)\in\mathcal{J}} \{\mathcal{X} \cup \{(\tau, u, \lambda^-)\}\}\Big)$$

and go to step *III*.

Step*IX*: Save the set of sequences $\mathcal{X}$ as candidate solution, i.e.:

$$\mathcal{C} = \mathcal{C} \cup \{\mathcal{X}\},$$

and go to step *III*.

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CHAPTER 4

Product Innovation with Lumpy Investments

Abstract This chapter considers a firm that has the option to undertake product innovations. For each product innovation the firm has to install a new production plant. We find that investments are larger and occur in a later stage when more of the old capital stock needs to be scrapped. Moreover, we obtain that the firm's investments increase when the technology produces more profitable products. We see that the firm in the beginning of the planning period adopts new technologies faster as time proceeds, but later on the opposite happens. Furthermore, we find that the firm does not invest when marginal profit (with respect to capital) becomes zero, but invests when marginal profit is negative. Moreover, we find that if the time it takes to double the efficiency of technology is larger than the time it takes for the capital stock to depreciate, the firm undertakes an initial investment. Finally, we show that, when demand decreases over time and when fixed investment cost is higher, the firm invests less throughout the planning period, the time between two investments increases and the first investment is delayed.

4.1 Introduction

In today's knowledge economy innovation is of prime importance. Innovation has led to extraordinary productivity gains in the 1990s. In current business practice it is felt that the heat is on and that firms must innovate faster just to stand still (The Economist, October 13th 2007, Innovation: Something new under the sun). Therefore, technological progress is a crucial input for firms in taking their investment decisions. Greenwood et al. (1997) argue that technological progress is the main driver of economic growth. They discovered that in the post-war period in the US about 60% of labor productivity growth was investment specific. Yorukoglu (1998) notes that information technology is a prime example where embodied technological progress led to an improvement of computing technology on the order of 20 times within less than a decade in the 1980s-90s.

This chapter combines technology adoption with capital accumulation, taking into account technological progress. The aim of this chapter is to study the decision of when to introduce a new product. To do so we employ the Impulse Control modeling approach that is perfectly suitable to take into account the disruptive changes caused by innovations. This also enables us to determine the length of the time interval that the firm uses a particular technology, when it is time to launch a new product generation, and how these decisions interact with the firm's capital accumulation behavior. In Kort (1989) a dynamic model of the firm is designed in which capital stock jumps upward at discrete points in time at which the firm invests. However, technological progress in not taken into account.

An example where a firm has to decide about investments in new generations of products is the LCD industry. With every new generation the size of the mother glass or substrate increases. As the LCD panels are cut out of the substrate, the substrate on the one hand determines which panel sizes can be produced and on the other hand how efficient each possible panel size can be produced. We have a process innovation, because a larger glass area provides a more efficient solution of the "cutting problem", and thus lower costs in the production process. A product innovation arises, because the larger area of the substrate makes it possible to produce larger screens. For a firm it is important to determine when it is optimal to introduce a new product. However, since the new product will decrease the demand of the old product, the moment of introduction is crucial.

Feichtinger et al. (2006) employs a vintage capital goods structure to study the effect of embodied technological progress on the investment behavior of the firm. They show that in the case that a firm has market power a negative anticipation effect occurs, i.e. when technological progress goes faster in the future, it is optimal for the firm to decrease investments in the current generation of capital goods. However, a direct implication of the vintage capital approach is that the firm adopts an infinite amount of different technologies. Of course, in practice a firm can adopt a new technology a limited number of times.

Grass et al. (2012) also combines technology adoption with capital accumulation, while taking into account technological progress. They find that investment jumps upward right at the moment that a new technology is adopted, and that the larger the firm the later the investment in a new technology takes place. Moreover, they find that when a firm has market power, the firm cuts down on investment before a new technology is adopted. Whereas Grass et al. (2012) limits itself to process innovation, we concentrate on studying product innovation. Grass et al. (2012) use a Multi-Stage optimal control approach where a firm adopts a new technology in each stage. Unlike Feichtinger et al. (2006), the number of technology adoptions is limited. However, the number of innovations is not determined by the model, but fixed exogenously instead. Unlike Feichtinger et al. (2006) and Grass et al. (2012), in this paper capital accumulation only occurs in lumps. Moreover, these lumps are determined by the model, i.e. the lumpy investments are endogenous. In Saglam (2011) a multi-stage optimal control model is studied where the number of technology adoptions are endogenous. However, unlike our paper, the model does not incorporate any (fixed) cost associated with the adoption and the considered firm has no market power. In Boucekkine et al. (2004) a two-stage optimal control model is considered, where only one adoption occurs, without adoption (fixed) cost. Both Boucekkine et al. (2004) and Saglam (2011) incorporate learning, were the firm raises productivity of a given technology over time due to learning and revenue is linear in the capital stock.

This chapter is comparable with Grass et al. (2012). However, unlike Grass et al. (2012), we do not need to fix the number of technology adoptions beforehand and we do incorporate a (fixed) cost associated with this adoption. When dealing with product innovation, firms do not always have to scrap all capital goods. Sometimes measures are taken to allow new or updated parts to be fitted to old or outdated assemblies. As in Grass et al. (2012), we can model all situations in between the extreme cases where after every new investment the old capital goods are scrapped and the case where all the capital goods can be kept after adopting a new technology.

The method used to study firm behavior in this paper is Impulse Control. Impulse Control theory is a variant of optimal control theory where discontinuities (i.e. jumps) in the state variable are allowed. In Impulse Control the moments of these jumps as well as the sizes of the jumps are decision variables. Blaquière (1977a; 1977b; 1979; 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so-called Impulse Control Maximum Principle, that gives necessary and sufficient optimality conditions for solving such problems. Blaquière's Impulse Control analysis is based on the present value Hamiltonian form. In this chapter we apply the Impulse Control theorem in the current value Hamiltonian framework as derived in Chahim et al. (2012).

One of the striking results is that the firm does not invest when the marginal profit (with respect to capital) becomes zero, but invests when marginal profit is negative. Furthermore, we obtain that the firm in the beginning of the planning period adopts new technologies faster as time proceeds, but after some moment in time later technologies are used for a longer time period. This behavior is different from Grass et al. (2012), who finds that the firm adopts new technologies faster as time proceeds for the whole planning period, but this also differs from the results found in Saglam (2011), who finds that later technologies are used during a longer time period. Our results are somehow a combination of both. Moreover, we find that if the time it takes to double the efficiency of technology is larger than the time it takes for the capital stock to depreciate, the firm undertakes an initial investment. Finally, we show that when demand decreases over time the firm invest less throughout the planning period and that the first investment is delayed.

This chapter is organized as follows. In Section 4.2 we give the general setting and build up the Impulse Control model. Section 4.3 derives the necessary optimality conditions, whereas Section 4.4 gives a brief description of the algorithms present in the literature dealing with the Impulse Control Maximum Principle. In Section 4.5 we study the investment behavior of the firm, and in Section 4.6 we extend this analysis by adding decreasing demand, i.e. demand decreases over time due to competitors producing better products because of technological progress. Finally, in Section 4.7 we conclude and give some recommendations for future research.

4.2 The Model

Consider a firm that invests in lumps over time. Each time it invests it installs a production plant suitable to produce the new product. Due to product innovation the quality of the products, and thus also demand, increases over time. This implies that the later an investment takes place, the better products can be made due to these investments.

This is formalized as follows. A plant being installed at time τ will make products from which the price is given by the following inverse demand function:

$$p(t) = \theta(\tau) - q(t), \quad \text{for} \quad t \geq \tau,$$

where $q(t)$ is the output at time t and $\theta(\tau) = 1 + b\tau$ is the state of technology that the firm adopts at time τ[1]. We further assume that technology within the firm does not change between two technology adoptions, i.e. $\dot{\theta}(t) = 0$ for all $t \neq \tau$. At the moment the firm adopts a technology, the firm's technology change is given by

$$\theta(\tau_i^+) - \theta(\tau_i^-) = 1 + b\tau_i - \theta(\tau_i^-).$$

Hence, as in Feichtinger et al. (2006) and Grass et al. (2012) we impose that technological progress increases linearly over time, where b is a positive constant. In Saglam (2011) technology increases exponentially over time and in Boucekkine et al. (2004) there are only two different technologies available. We assume a simple production function in the sense that one capital good produces one unit of output. Denoting the stock of capital goods by $K(t)$, this gives

$$K(t) = q(t).$$

We impose that only the capital stock of the new plant is able to produce the new products, i.e. each plant has its own capital stock that produces the products with a quality associated with the timing of the investment in this plant. In this setting we can also model a situation where just $100\gamma\%$, where $\gamma \in [0, 1]$, of the capital stock is scrapped, while the remaining machines or tools can be reused for the new product. Hence, full scrapping corresponds to the case where $\gamma = 1$. This implies that old products, and thus also old capital goods, become worthless after the new plant is installed, implying that the old capital goods can be scrapped.

Denoting investment by $I(t)$, at the moment the firm invests (adopts a new technology) capital stock changes by

$$K(\tau^+) - K(\tau^-) = I(\tau) - \gamma K(\tau^-).$$

At time zero the capital stock is equal to zero, i.e.

$$K(0) = 0.$$

For each plant it holds that capital stock depreciates with rate δ, i.e.

$$\dot{K}(t) = -\delta K(t).$$

Investing in a plant implies that the firm has to pay a fixed cost, i.e. part of the cost is independent of the plant size, and a variable cost that more than proportionally

[1] We assume that technology is continuously changing with rate b. However, the technology within the firm is the technology that the firm adopts at time τ.

increases with the size of the plant. In particular, we assume that the investment cost is given by

$$C(I) = \begin{cases} C + \alpha I + \beta I^2 & \text{for } I > 0 \\ 0 & \text{for } I = 0. \end{cases}$$

This type of investment cost function, without the fixed cost, is common in the literature (e.g., among others, see Grass et al. (2012), Sethi and Thompson (2006) and Seierstad and Sydsæter (1987)), where besides the fixed cost, the linear term consists of acquisition cost, where the unit price is equal to α and the quadratic term represents the adjustment cost. In "ordinary" optimal control the investment cost function does not include a fixed cost, because this violates the continuity of the cost function with respect to its arguments, i.e. the control and the state variable.

Total discounted revenue in $[0, T]$ is given by

$$\int_0^T e^{-rt} \left[\theta\left(t\right) - K\left(t\right)\right] K(t) dt,$$

where revenue is determined by output price times output. Since we have a finite time planning period, a salvage value has to be defined. This salvage value is equal to the value of the firm at the time horizon T. We assume that this value is given by

$$e^{-rT} \frac{[\theta(\tau_N) - K(T^+)]K(T^+)}{r + \delta},$$

where τ_N denotes the time of the last investment. The salvage value (4.1) is a lower bound of the discounted revenue stream of the firm after the planning period.

Total discounted investment cost are given by the sum of the cost of adopting a new technology, discounted at the time the adoption takes place, i.e. $\tau_1, \ldots \tau_N$, with $0 \leq \tau_1 < \tau_2 \ldots < \tau_N \leq T$. This results in

$$\sum_{i=1}^{N} e^{-r\tau_i} \left(C + \alpha I\left(\tau_i\right) + \beta I\left(\tau_i\right)^2\right).$$

The above gives rise to the following impulse control model:

$$\begin{aligned} &\max_{I, \tau_i, N} \int_0^T e^{-rt} \left[\theta\left(t\right) - K\left(t\right)\right] K(t) dt \\ &- \sum_{i=1}^{N} e^{-r\tau_i} \left(C + \alpha I\left(\tau_i\right) + \beta I\left(\tau_i\right)^2\right) \\ &+ e^{-rT} \frac{[\theta(\tau_N) - K(T^+)]K(T^+)}{r + \delta} \end{aligned} \tag{4.1}$$

subject to

$$
\begin{aligned}
\dot{K}(t) &= -\delta K(t), && \text{for} \quad t \neq \tau_i \quad (i = 1, ..., N), && (4.2)\\
\dot{\theta}(t) &= 0, && \text{for} \quad t \neq \tau_i \quad (i = 1, ..., N), && (4.3)\\
K(\tau_i^+) - K(\tau_i^-) &= I(\tau_i) - \gamma K(\tau_i^-), && \text{for} \quad i = 1, ..., N, && (4.4)\\
\theta(\tau_i^+) - \theta(\tau_i^-) &= 1 + b\tau_i - \theta(\tau_i^-), && \text{for} \quad i = 1, ..., N, && (4.5)\\
K(0) &= 0, && && (4.6)\\
\theta(0) &= 1. && && (4.7)
\end{aligned}
$$

This is an Impulse Control problem as described in Blaquière (1977a; 1977b; 1979; 1985). Note that this innovation model only contains an impulse control variable and no ordinary control variable. This approach differs from the multi-stage approach used in Grass et al. (2012), because here investment takes place in lumps and every investment goes along with a fixed cost. As in Grass et al. (2012) we can model all situations between the extreme cases where after every new investment the old capital goods are scrapped ($\gamma = 1$) and where all the capital can be kept ($\gamma = 0$) to produce the new product. Another benefit of the above model is that the number of technology changes are endogenous.

4.3 Necessary Optimality Conditions

We apply the impulse control maximum principle in current value formulation derived in Chahim et al. (2012). Other references deriving the necessary optimality conditions for the Impulse Control problems are Blaquière (1977a; 1977b; 1979; 1985), Seierstad (1981) and Seierstad and Sydsæter (1987). We define the Hamiltonian *Ham* and the Impulse Hamiltonian *IHam*

$$
\begin{aligned}
Ham(\theta, K, \lambda_1, t) &= [\theta - K]K - \lambda_1 \delta K, && (4.8)\\
IHam(K, I, \lambda_1, \lambda_2, t) &= -C - \alpha I - \beta I^2 + \lambda_1(I - \gamma K) && (4.9)\\
&\quad +\lambda_2 (1 + bt - \theta)),
\end{aligned}
$$

and obtain the adjoint equations

$$
\begin{aligned}
\dot{\lambda}_1(t) &= (r + \delta)\lambda_1(t) - \theta(t) + 2K(t), \quad \text{for} \quad t \neq \tau_i, \quad i = 1, \ldots, N, && (4.10)\\
\dot{\lambda}_2(t) &= r\lambda_2(t) - K(t), \quad \text{for} \quad t \neq \tau_i, \quad i = 1, \ldots, N. && (4.11)
\end{aligned}
$$

The jump conditions at $t = \tau_i$ $(i = 1, \ldots, N)$ are

$$
\begin{aligned}
-\alpha - 2\beta I(\tau_i) + \lambda_1\left(\tau_i^+\right) &= 0, && (4.12)\\
\lambda_1\left(\tau_i^+\right) - \lambda_1\left(\tau_i^-\right) &= \gamma \lambda_1\left(\tau_i^+\right), && (4.13)\\
\lambda_2\left(\tau_i^+\right) - \lambda_2\left(\tau_i^-\right) &= \lambda_2\left(\tau_i^+\right), && (4.14)
\end{aligned}
$$

from which we conclude that

$$\lambda_1\left(\tau_i^-\right) = (1-\gamma)\,\lambda_1\left(\tau_i^+\right),$$

which equals zero for $\gamma = 1$, and

$$\lambda_2\left(\tau_i^-\right) = 0.$$

The condition for determining the optimal switching time τ_i is

$$\begin{aligned}
&\left[\theta\left(\tau_i^+\right) - K\left(\tau_i^+\right)\right] K\left(\tau_i^+\right) - \left[\theta\left(\tau_i^-\right) - K\left(\tau_i^-\right)\right] K\left(\tau_i^-\right)\\
&-\lambda_1\left(\tau_i^+\right)\delta K\left(\tau_i^+\right) + \lambda_1\left(\tau_i^-\right)\delta K\left(\tau_i^-\right) - rC - r\alpha I\left(\tau_i\right) - r\beta I\left(\tau_i\right)^2 - b\lambda_2\left(\tau_i^+\right)\\
&\begin{cases} > 0 & \text{for } \tau_i = 0\\ = 0 & \text{for } \tau_i \in (0,T)\\ < 0 & \text{for } \tau_i = T.\end{cases}
\end{aligned} \tag{4.15}$$

The transversality conditions are

$$\lambda_1\left(T^+\right) = \frac{\theta(\tau_N) - 2K\left(T^+\right)}{r+\delta}, \tag{4.16}$$

and

$$\lambda_2\left(T^+\right) = \frac{K(T^+)}{r+\delta}. \tag{4.17}$$

At the non-jump points $t \neq \tau_I$ $(i = 1,, ..., N)$ it holds that $\lim_{I\to 0}\frac{\partial IHam}{\partial I} = \infty$ due to the fixed cost. Hence, the conditions for applying the Impulse Control Maximum Principle are met, see Section 2.2.

4.4 Algorithm

In the literature three different algorithms are derived based on the Impulse Control Maximum Principle (Blaquière (1977a; 1977b; 1979; 1985) and Chahim et al. (2012)). Luhmer (1986) derived a forward algorithm (starts at time 0) and Kort (1989, pp. 62–70) derived a backward algorithm (starts at final time horizon T). Luhmer (1986) starts at $t = 0$ and uses the costate variable, as input to initialize his algorithm. Kort (1989) implements a backward algorithm that starts at the time horizon, i.e. $t = T$, and initializes the algorithm using the values of the state variables. Finally, Grass and Chahim (2012) design an algorithm that is a combination of continuation techniques and a (multipoint) Boundary Value Problem (BVP) to solve Impulse Control problems (see Chapter 5).

The problem described by (4.1)–(4.7) has two state variables, the stock of capital $K(t)$ and technology $\theta(t)$. The question is which algorithm is most suitable for this model. Applying the forward algorithm to problem (4.1)–(4.7) has a drawback. Namely, we have to guess the initial values for the two costate variables, $\lambda_1(0)$ and $\lambda_2(0)$. A wrong guess of the costate variables at the initial time results in a solution that does not satisfy the transversality conditions (4.16) and (4.17), which implies that the necessary optimality conditions are not satisfied. For the backward algorithm we start with choosing values for the state variables at time T. The resulting solution always satisfies the necessary optimality conditions, but here the problem is that the algorithm has to end up at the right $K(0)$. In other words, with the backward algorithm one can apply the right necessary conditions to the wrong problem. An example where the backward algorithm is applied successfully can be found in Chapter 3. Moreover, in Chapter 3 clear upper and lower bounds have been derived for the state variable.

In addition, the backward algorithm has another drawback. When we apply it to the problem described by (4.1)–(4.7), starting at the time horizon and going back in time requires knowledge of the technology before the investment. In particular, we obtain from equation (4.15) that we need to know $\theta(\tau_N^+) = 1 + b\tau_N$ and $\theta(\tau_N^-) = \theta(\tau_{N-1}) = 1 + b\tau_{N-1}$. Hence, solving this equation for τ_N requires that we know τ_{N-1}. And with the backward algorithm, this predecessor is not known. We conclude that the backward algorithm is not suitable to solve our model as presented in this chapter.

The third algorithm described in the literature is an algorithm that considers the problem described by (4.1)–(4.7) as a (multipoint) Boundary Value Problem (BVP) and uses a continuation technique to solve it. The main idea behind the algorithm is as follows. To find the solution of the problem described by (4.1)–(4.7) we can apply a continuation strategy with respect to the time horizon T, i.e. T is our continuation variable. The algorithm for this approach is described in Box 4.1. To initialize the algorithm, the problem is solved for $T = 0$. Assuming that a unique solution exists for $T = 0$, the initial conditions together with the transversality conditions combined with the necessary conditions results in a set of n equation with n unknowns. Given a solution for $T = 0$, T is increased (continued) during the continuation process whereas the conditions for possible jumps are monitored. If the conditions for a jump are satisfied, the BVP is adapted to this situation. With this new solution the continuation is pursued. In Chapter 5 this algorithm more extensively. In Grass (2012) also a BVP approach in combination with continuation is described, but that paper the focus is on ordinary optimal control problems.

Define T as time horizon for the problem.
Define $\bar{T}$ to be a continuation variable.
Set $\bar{T} = 0$ and $\tau_l = 0$.

Step 1: Find jump in $[\tau_l, \bar{T}]$ for:

case 1: A jump occurs at the end, i.e. at $t = \bar{T}$, save as JumpSol.

case 2: No jump at the end, save as noJumpSol.

Step 2: Start the continuation for $\bar{T} \in (\tau_l, T)$ with JumpSol until interior jump condition is satisfied, i.e. let $\bar{T}$ increase until (4.12)–(4.15) are satisfied for some $t = \tau$. Set $\tau_l = \tau$, save as JumpSol. Also continue the result of noJumpSol until $\bar{T} = \tau_l$, save as noJumpSol. If $\bar{T} \geq T$ without satisfying interior jump conditions, stop.

case 1: Objective of JumpSol higher than objective no-JumpSol, add arc and go to step 1.

case 2: Objective of JumpSol is lower than objective no-JumpSol, go to step 3.

Step 3: Continue the solution of noJumpSol until the interior jump condition (4.12)–(4.15) is satisfied for $t = \tau \in (\tau_l, T)$. Set $\tau_l = \tau$, save as JumpSol, add arc, and go to step 1. If $\bar{T} \geq T$ without satisfying interior jump conditions, stop.

Box 4.1 – (Multipoint) BVP and continuation algorithm

4.5 Endogenous Lumpy Investments

When a firm is dealing with market power, the output price decreases with the quantity that is produced. Since it holds in this model that with one unit of capital stock one unit of output is produced, we have that the output price decreases with the amount of capital. So during the time period between two investments the output price increases, since depreciation decreases capital stock. We consider no scrapping, partial scrapping and total scrapping, i.e. we consider $\gamma = 0$, $\gamma = 0.5$ and $\gamma = 1$. We provide a numerical analysis starting with the parameter values

$$b = \frac{1}{n}\log 2 = \frac{1}{2}\log 2, \quad \alpha = 0, \quad \beta = 0.2, \quad C = 2 \quad r = 0.04, \quad \delta = 0.2,$$

which we consider as the benchmark throughout this chapter. As in Grass et al. (2012), we base our value for b on Moore's law[2], where the value for b is such that the efficiency of the technology doubles every n years where we put $n = 2$. The results of the first ten investments, are presented in Table 4.1 for $T = 100$. Table 4.6 of Appendix 4A presents all investments up until $T = 100$.

Ignoring the first and last investment, we see that the better the technology is, the larger the investment becomes. It seems as if the firm delays the first investment (compared to the others) to start production of a new good. In Figure 4.1a this is clearly seen (also see Figure 4.4a and Figure 4.6a in Appendix 4A). To understand what happens with the first investment we have to distinguish between $\gamma < 1$ and $\gamma = 1$. When $\gamma < 1$ capital growth is increased with each investment without fully scrapping the old capital stock. Because there is only limited scrapping, at an early stage the firm undertakes a relatively high investment to start production. A firm only undertakes this relatively high investment if there is limited scrapping, because the investments help to increase the capital stock in the future. This behavior is clearly seen in Figure 4.1a. Connecting the points in Figure 4.1a ignoring the first and last investment not only tells us that the first investment is relatively large, but also that the last investment is small. This last investment being small occurs due to the fact that the salvage value of the problem is (too) low, because it does not take into account technological improvement after time T.

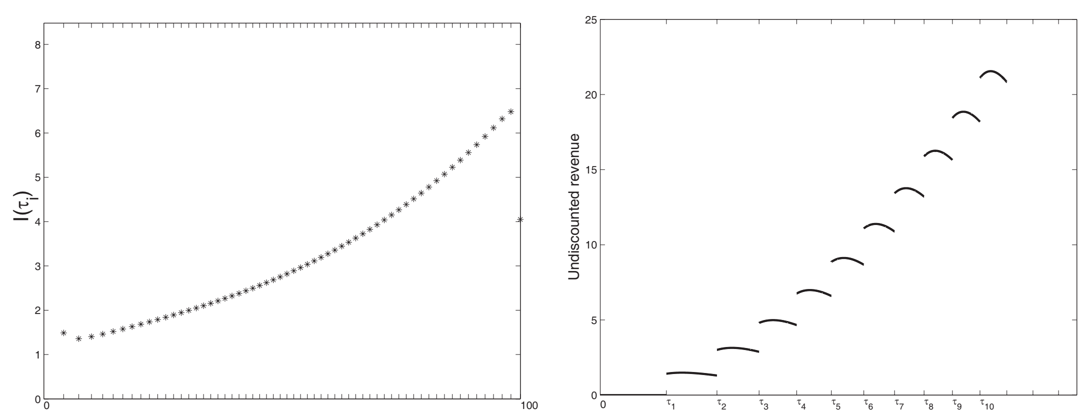
Table 4.1 shows that the higher the scrapping percentage the larger the investments become. This makes sense because a firm that wants similar production has to invest extra to replace the scrapped parts. This scrapping increases the investment cost and at the same time, due to the quadratic term in the investment cost function, investing such that the same level of capital is reached as in the case of no scrapping, is too expensive. Hence, the optimal level of capital stock in the case of scrapping is lower than under no scrapping, which explains the lower revenue. Table 4.6 of Appendix 4A presents all investments up until $T = 100$ (Table 4.7, 4.8 and 4.9 present full results for $\gamma = 0$, $\frac{1}{2}$ and 1, respectively). It shows that a higher scrapping percentage decreases the number of investments during the planning pe-

[2]Moore's law still holds, The Economist, July 14th 2012, Chipping in: A deal to keep Moore's law alive.

	$\gamma = 0$	$\gamma = 0.5$	$\gamma = 1$
$(\tau_i : I)$	4.1651 : 1.4877	4.1462 : 1.4682	3.8509 : 1.3689
	7.3464 : 1.3571	7.4147 : 1.7204	7.1308 : 1.9589
	10.0022 : 1.4032	10.1649 : 2.0101	9.9511 : 2.4614
	12.3693 : 1.4610	12.6433 : 2.2785	12.5559 : 2.9262
	14.5474 : 1.5188	14.9499 : 2.5312	15.0389 : 3.3716
	16.5895 : 1.5751	17.1370 : 2.7731	17.4476 : 3.8067
	18.5276 : 1.6299	19.2361 : 3.0070	19.8100 : 4.2365
	20.3835 : 1.6837	21.2682 : 3.2353	22.1437 : 4.6639
	22.1724 : 1.7365	23.2479 : 3.4594	24.4606 : 5.0910
	23.9056 : 1.7887	25.1861 : 3.6805	26.7688 : 5.5191
Rev	802.4809	790.1920	771.3955
ICost	35.3109	67.8103	97.6050
Profit	767.1700	722.3817	673.7904

Table 4.1 – First ten investments of Impulse Control solutions for different γ, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$. $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

riod. Another striking effect can be noticed when looking at Figure 4.1b. We see



(a) Lumpy investments, $I(\tau_i)$.

(b) Undiscounted revenue for the first ten investments.

Figure 4.1 – Lumpy investments and undiscounted revenue, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

that the firm invests in a new product such that marginal revenue (with respect to capital) is negative. In a "static" model (i.e. a model that does not depend on time) we know that the firms optimize profit and hence invest at the moment that marginal cost is equal to marginal revenue. Since we did not include any operation cost, we know that marginal cost is equal to zero. Hence, when marginal revenue is equal to zero, (i.e. $K(\tau) = \theta(\tau)/2$) investment would be optimal according to this rule. In our dynamic setting it is impossible to stay at the point where marginal revenue is equal to zero, due to depreciation. In Table 4.2 we show the results for a case where we have no depreciation. We see that indeed the investments are such that the level of capital is set to $K(\tau) = \theta(\tau)/2$. In the case that we have depreciation, the firm overinvests, i.e., invests such that marginal revenue is negative. Then up until the next investment, marginal revenue increases, becomes zero after some time, and then turns positive. In Figure 4.2 we have plotted the length of the time interval

τ_i	$\theta(\tau_i^+)$	$K(\tau_i^+)$	$\frac{\theta}{K}$
19.6234	7.8009	3.8574	2.0224
34.5329	12.9682	6.4650	2.0059
50.7184	18.5777	9.2706	2.0039
70.6244	25.4766	12.7165	2.0034
99.7453	35.5691	17.7443	2.0045

Table 4.2 – Technology level and capital, where $T = 100$, $\gamma = 0$, $r = 0.04$, $\delta = 0$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

between two investments. We see that in the beginning of the planning period the firm adopts new technologies faster as time proceeds and after some moment it uses later technologies for a longer time period. This behavior is different from Grass et al. (2012), who finds that the firm adopt new technologies faster as time proceeds for the whole planning period, but this also differs from the results found in Saglam (2011), who finds that later technologies are used during a longer time period. Our results are somehow a combination of both. An explanation for this could be that the firm in the beginning of the planning period does not invest much since productivity is low. After some time technological progress is such that each investment is more profitable, which causes the corresponding capital goods are used for a longer time. For this reason the time between investments increases. Also for higher T a similar effect is found.

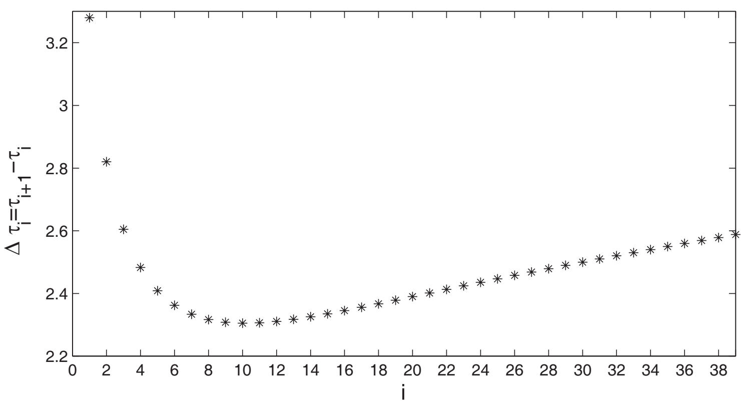


Figure 4.2 – The length between two investments, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

4.5.1 Sensitivity Analysis with Respect to the Rate of Technology Change

Here we study how the rate of technological progress affects the investment behavior of a firm. Remember that we have assumed, using Moore's law, that efficiency of technology doubles every n years, setting $n = 2$ for our benchmark case. Table 4.3 shows the first ten investments for different values of the technology rate b. For all investments up until $T = 100$ see Table 4.10 of Appendix 4A (or Table 4.11-4.15 for each level of technology change separately). When $n > 5$ an investment takes place at $t = 0$. The explanation behind this is that for $n > 5$ we have, under Moore's law, that it takes more than five years for the efficiency of a technology to double. Since we have a depreciation rate of 20%, this means it takes more time to double the efficiency of a technology than the capital stock to depreciate to half of its original level. So the firm has no incentive to wait and invests at $t = 0$.

4.5.2 Sensitivity Analysis with Respect to the Fixed Cost

One of the main differences between Grass et al. (2012), Boucekkine et al. (2004) and Saglam (2011) is that they do not incorporate any (fixed) cost and this chapter assumes that a fixed cost is included for each investment. Here we study how increasing these fixed cost affects the investment behavior of a firm. Table 4.4 shows the first ten investments for each size of fixed cost. For all investments up until $T = 100$ see Table 4.16 of Appendix 4A (or Table 4.17-4.20 for each for each size of fixed cost separately). It is easily seen, that if we increase the fixed cost, the first investment is delayed and at the same time the time period between two investments increases. Hence, the number of investments decreases if the fixed cost increase. Comparing

	$b = \frac{1}{3}\log 2$	$b = \frac{1}{4}\log 2$	$b = \frac{1}{5}\log 2$	$b = \frac{1}{6}\log 2$	$b = \frac{1}{10}\log 2$
$(\tau_i : I)$	4.6759 : 1.3116	5.1658 : 1.2381	5.5832 : 1.1914	0 : 0.7418	0 : 0.7752
	8.6561 : 1.5392	9.7814 : 1.4539	10.7534 : 1.3980	7.7656 : 1.2080	9.7219 : 1.1432
	1.9662 : 1.7807	13.5911 : 1.6692	14.9977 : 1.5949	13.0448 : 1.4152	16.3705 : 1.3132
	14.9229 : 1.9995	16.9755 : 1.8614	18.7535 : 1.7685	17.4932 : 1.5907	21.9534 : 1.4524
	17.6530 : 2.2025	20.0857 : 2.0378	22.1932 : 1.9266	21.4683 : 1.7459	26.9204 : 1.5730
	20.2231 : 2.3943	23.0005 : 2.2031	25.4066 : 2.0736	25.1251 : 1.8873	31.4676 : 1.6810
	22.6732 : 2.5779	25.7678 : 2.3601	28.4478 : 2.2125	28.5485 : 2.0188	35.7025 : 1.7797
	25.0300 : 2.7553	28.4191 : 2.5108	31.3530 : 2.3450	31.7914 : 2.1429	39.6921 : 1.8712
	27.3121 : 2.9280	30.9766 : 2.6566	34.1472 : 2.4725	34.8894 : 2.2610	43.4813 : 1.9569
	29.5335 : 3.0970	33.4569 : 2.7986	36.8494 : 2.5960	37.8681 : 2.3745	47.1023 : 2.0376
Rev	371.5616	220.0775	148.0959	108.6965	47.4170
ICost	39.2258	27.6829	21.7123	19.5772	12.9673
Profit	332.3358	192.3946	126.3837	89.1193	34.4497

Table 4.3 – First ten investments of Impulse Control solutions for different b, where $T = 100$, $\gamma = 0.5$, $r = 0.04$. $\delta = 0.2$, $\beta = 0.2$, $\alpha = 0$. $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

the results more carefully, we see that the size of the lumpy investments (i.e. jumps) increases when the fixed cost increases.

	$C = 4$	$C = 8$	$C = 16$	$C = 32$
$(\tau_i : I)$	5.7915 : 1.8832	8.0844 : 2.4856	11.1517 : 3.3199	15.2866 : 4.4754
	9.6593 : 2.2099	12.7147 : 2.9206	16.6712 : 3.8947	21.8148 : 5.2241
	12.8816 : 2.5607	16.5386 : 3.3546	21.1933 : 4.4297	27.1293 : 5.8789
	15.7638 : 2.8797	19.9372 : 3.7422	25.1901 : 4.8993	31.8052 : 6.4443
	18.4283 : 3.1763	23.0621 : 4.0984	28.8471 : 5.3256	36.0657 : 6.9513
	20.9394 : 3.4571	25.9923 : 4.4325	32.2606 : 5.7215	40.0266 : 7.4169
	23.3358 : 3.7265	28.7755 : 4.7502	35.4889 : 6.0947	43.7577 : 7.8515
	25.6433 : 3.9871	31.4435 : 5.0556	38.5705 : 6.4506	47.3050 : 8.2618
	27.8799 : 4.2412	34.0186 : 5.3513	41.5327 : 6.7926	50.7012 : 8.6525
	30.0590 : 4.4903	36.5173 : 5.6394	44.3957 : 7.1237	53.9701 : 9.0270
Rev	780.7835	769.1875	747.0746	712.6433
ICost	79.5936	96.8939	120.5584	150.9987
Profit	701.1899	672.2936	626.5162	561.6447

Table 4.4 – Impulse Control solutions for different C, where $T = 100$, $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

4.6 Lumpy Investments under Decreasing Demand

In this section we consider the case where the demand for an existing product decreases over time. A main reason could be that the competitors' products become better due to their product innovations. We incorporate decreasing demand by setting $\dot{\theta}(t) = -\eta t$, where η is some positive constant. Since it is reasonable to assume $\delta > \eta > 0$,[3] the output price after investment is first increasing and then decreasing, see Figure 4.3. Hence, if a firm invests, capital stock depreciates and the output price increases, and after some time this output price is decreasing due to this decreasing demand. Then the model becomes

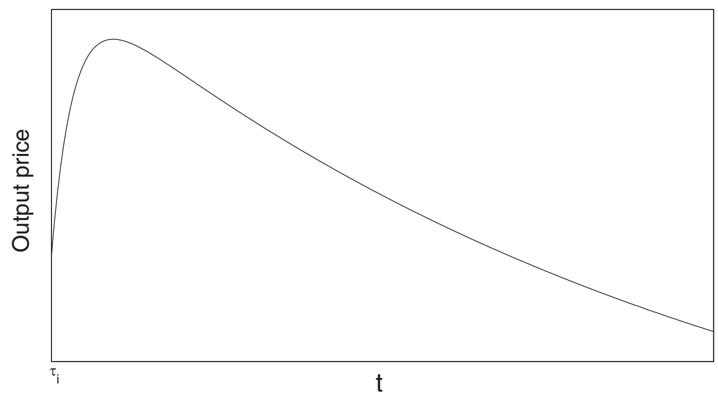


Figure 4.3 – Output price as a function of time for $\delta > \eta$.

$$\begin{aligned}
&\max_{I,\tau,N} \int_0^T e^{-rt} \left[\theta\left(t\right) - K\left(t\right)\right] K(t)dt \\
&- \sum_{i=1}^{N} e^{-r\tau_i} \left(C + \alpha I\left(\tau_i\right) + \beta I\left(\tau_i\right)^2\right) \\
&+ e^{-rT} \frac{\left[\theta\left(T^+\right) - K(T^+)\right]K(T^+)}{r+\delta+\eta},
\end{aligned} \tag{4.18}$$

[3] Since we are dealing with product innovation and assume a depreciation rate of 20% it is unlikely that demand decreases by more than (or equal to) 20% and hence we do not consider $\eta \geq \delta > 0$.

subject to

$$\begin{aligned}
\dot{K}(t) &= -\delta K(t), && \text{for} \quad t \neq \tau_i \quad (i = 1, ..., N), && (4.19)\\
\dot{\theta}(t) &= -\eta\theta(t), && \text{for} \quad t \neq \tau_i \quad (i = 1, ..., N), && (4.20)\\
K(\tau_i^+) - K(\tau_i^-) &= I(\tau_i) - \gamma K(\tau_i^-), && \text{for} \quad i = 1, ..., N, && (4.21)\\
\theta(\tau_i^+) - \theta(\tau_i^-) &= 1 + b\tau_i - \theta(\tau_i^-), && \text{for} \quad i = 1, ..., N, && (4.22)\\
K(0) &= 0, && && (4.23)\\
\theta(0) &= 1. && && (4.24)
\end{aligned}$$

Recall that in Section 4.5 the output price was decreasing in capital. Hence, due to depreciation the output price is increasing in the time period between two investments. Since we are considering product innovation, it makes more sense that demand of a given product during the time period decreases. This is because over time new products are invented by other firms, which reduce demand of the current product. This demand decrease has a negative effect on output price and hence the firm has even a greater incentive to invest in a new technology.

Looking at the results of Table 4.5 and Table 4.21 (or Table 4.22-4.24 for each decay rate of the demand separately) we can see that a change in the decrease of demand directly affects the investment behavior. It is clear to see, that if we increase η the first investment is delayed and at the same time the time period between two investments also increases. Hence, the number of investments decreases if the decay rate of the demand increases. This makes sense, since less demand makes investing less attractive. This results in a lower investment cost for higher η. Moreover, the larger η the lower the output price (compared to a lower η) and hence the lower the revenue.

4.7 Conclusions and Recommendations

This chapter employs an Impulse Control modeling approach that is suitable to take into account the disruptive changes caused by innovations. We describe and implement an algorithm based on current value necessary optimality conditions. The necessary conditions are solved using a (multipoint) Boundary Value Problem (BVP) combined with some continuation techniques.

From an economic point of view we have derived some guidelines for lumpy investments in new technology:

- A striking result is that the firm does not invest when marginal profit (with

	$\eta = 0.01$	$\eta = 0.02$	$\eta = 0.03$
$(\tau_i : I)$	5.2730 : 1.7250	6.3504 : 1.9594	7.5126 : 2.2042
	8.9696 : 2.0366	10.4003 : 2.3175	11.902 : 2.6060
	12.0850 : 2.3821	13.8098 : 2.7062	15.5932 : 3.0359
	14.9011 : 2.7029	16.8941 : 3.0676	18.9345 : 3.4366
	17.5308 : 3.0067	19.7779 : 3.4110	22.0629 : 3.8188
	20.0327 : 3.2991	22.5261 : 3.7427	25.0493 : 4.1897
	22.4425 : 3.5837	25.1779 : 4.0670	27.9368 : 4.5542
	24.7835 : 3.8631	27.7594 : 4.3869	30.7539 : 4.9156
	27.0723 : 4.1393	30.2889 : 4.7047	33.5212 : 5.2765
	29.3212 : 4.4136	32.7803 : 5.0219	36.2541 : 5.6390
Rev	762.5966	733.2291	701.2148
ICost	61.1145	56.6083	52.6074
Profit	701.4821	676.6208	648.6074

Table 4.5 – First ten investments of Impulse Control solutions for different η, where $T = 100$, $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

respect to capital) is zero, but invests when marginal profit is negative. Indeed, due to depreciation capital stock decreases in between two investments, implying that marginal profit goes up there due to the decreasing returns to scale assumption. The implication is that during such an interval first marginal profit is negative, but then after a while it turns positive and this stays that way until it is time for the next investment.

- We find that investments are larger and the time between investments is larger when more of the old capital stock needs to be scrapped. If a change in technology permits the firm to keep, update and reuse part of its capital stock, the investments are smaller.

- A nontrivial result is the optimal timing of investments. We see that the firm in the beginning of the planning period adopts new technologies faster as time proceeds, but later on the opposite happens. Moreover, we obtain that the firm's investments increase when the technology produces more profitable products.

- The experiments show that if the time it takes to double the efficiency of a technology is larger than the time it takes for the capital stock to depreciate to half of its original level, the firm undertakes an initial investment.

- Further sensitivity results were provided for a scenario of decreasing demand. We find that when demand decreases over time and when fixed investment cost is higher, then the firm invests less throughout the planning period, the time between two investments increases and the first investment is delayed.

Interesting directions for further work would be to consider running cost in the model or to introduce a learning effect. Another possible extension would be to let the scrapping percentage depend on time.

Appendix 4A: Tables and Figures

	$\gamma = 0$	$\gamma = 0.5$	$\gamma = 1$
$(\tau_i : I)$	4.1651 : 1.4877	4.1462 : 1.4682	3.8509 : 1.3689
	7.3464 : 1.3571	7.4147 : 1.7204	7.1308 : 1.9589
	10.0022 : 1.4032	10.1649 : 2.0101	9.9511 : 2.4614
	12.3693 : 1.4610	12.6433 : 2.2785	12.5559 : 2.9262
	14.5474 : 1.5188	14.9499 : 2.5312	15.0389 : 3.3716
	16.5895 : 1.5751	17.1370 : 2.7731	17.4476 : 3.8067
	18.5276 : 1.6299	19.2361 : 3.0070	19.8100 : 4.2365
	20.3835 : 1.6837	21.2682 : 3.2353	22.1437 : 4.6639
	22.1724 : 1.7365	23.2479 : 3.4594	24.4606 : 5.0910
	23.9056 : 1.7887	25.1861 : 3.6805	26.7688 : 5.5191
	25.5920 : 1.8407	27.0909 : 3.8994	29.0742 : 5.9490
	27.2385 : 1.8924	28.9689 : 4.1168	31.3809 : 6.3813
	28.8508 : 1.9443	30.8252 : 4.3333	33.6920 : 6.8164
	30.4336 : 1.9964	32.6640 : 4.5493	36.0096 : 7.2545
	31.9908 : 2.0488	34.4889 : 4.7652	38.3355 : 7.6957
	33.5258 : 2.1018	36.3027 : 4.9814	40.6707 : 8.1403
	35.0416 : 2.1554	38.1081 : 5.1982	43.0162 : 8.5881
	36.5406 : 2.2098	39.9072 : 5.4157	45.3723 : 9.0393
	38.0252 : 2.2651	41.7019 : 5.6343	47.7396 : 9.4937
	39.4975 : 2.3214	43.4940 : 5.8541	50.1182 : 9.9514
	40.9591 : 2.3788	45.2849 : 6.0753	52.5083 : 10.4123
	42.4119 : 2.4374	47.0759 : 6.2982	54.9099 : 10.8763
	43.8573 : 2.4973	48.8684 : 6.5229	57.3230 : 11.3435
	45.2968 : 2.5586	50.6635 : 6.7495	59.7474 : 11.8136
	46.7317 : 2.6214	52.4621 : 6.9783	62.1832 : 12.2867
	48.1632 : 2.6859	54.2654 : 7.2094	64.6300 : 12.7627
	49.5925 : 2.7520	56.0743 : 7.4431	67.0879 : 13.2415
	51.0207 : 2.8200	57.8895 : 7.6793	69.5566 : 13.7231
	52.4488 : 2.8900	59.7121 : 7.9184	72.0359 : 14.2075
	53.8779 : 2.9620	61.5428 : 8.1606	74.5258 : 14.6945
	55.3089 : 3.0362	63.3824 : 8.4059	77.0260 : 15.1843
	56.7427 : 3.1127	65.2318 : 8.6546	79.5364 : 15.6766
	58.1804 : 3.1917	67.0917 : 8.9070	82.0568 : 16.1716
	59.6228 : 3.2732	68.9629 : 9.1631	84.5872 : 16.6692

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	$\gamma = 0$	$\gamma = 0.5$	$\gamma = 1$
$(\tau_i : I)$	61.0707 : 3.3574	70.8463 : 9.4233	87.1274 : 17.1693
	62.5251 : 3.4445	72.7426 : 9.6878	89.6772 : 17.6720
	63.9868 : 3.5347	74.6526 : 9.9569	92.2367 : 18.1774
	65.4567 : 3.6279	76.5773 : 10.2307	94.8058 : 18.6853
	66.9357 : 3.7246	78.5174 : 10.5097	97.3844 : 19.1959
	68.4245 : 3.8248	80.4738 : 10.7940	99.9725 : 17.0969
	69.9242 : 3.9287	82.4476 : 11.0841	
	71.4355 : 4.0366	84.4396 : 11.3803	
	72.9593 : 4.1486	86.4508 : 11.6829	
	74.4967 : 4.2650	88.4824 : 11.9925	
	76.0484 : 4.3860	90.5354 : 12.3093	
	77.6154 : 4.5120	92.6110 : 12.6340	
	79.1987 : 4.6431	94.7105 : 12.9668	
	80.7994 : 4.7798	96.8355 : 13.2846	
	82.4183 : 4.9222	99.0358 : 10.8535	
	84.0566 : 5.0708		
	85.7154 : 5.2260		
	87.3959 : 5.3881		
	89.0991 : 5.5576		
	90.8264 : 5.7349		
	92.5790 : 5.9206		
	94.3584 : 6.1152		
	96.1659 : 6.3181		
	98.0055 : 6.4796		
	99.9896 : 4.0490		
Rev	802.4809	790.1920	771.3955
ICost	35.3109	67.8103	97.6050
Profit	767.1700	722.3817	673.7904

Table 4.6 – Impulse Control solutions for different γ, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
4.1651	1.4877	0	1.4877	2.4435
7.3464	1.3571	0.7874	2.1445	3.5461
10.0022	1.4032	1.2608	2.6640	4.4665
12.3693	1.4610	1.6593	3.1204	5.2869
14.5474	1.5188	2.0184	3.5372	6.0418
16.5895	1.5751	2.3512	3.9263	6.7495
18.5276	1.6299	2.6646	4.2946	7.4212
20.3835	1.6837	2.9629	4.6466	8.0644
22.1724	1.7365	3.2490	4.9855	8.6844
23.9056	1.7887	3.5250	5.3138	9.2851
25.5920	1.8407	3.7925	5.6332	9.8695
27.2385	1.8924	4.0526	5.9451	10.4402
28.8508	1.9443	4.3064	6.2507	10.9989
30.4336	1.9964	4.5546	6.5510	11.5475
31.9908	2.0488	4.7979	6.8467	12.0872
33.5258	2.1018	5.0368	7.1386	12.6192
35.0416	2.1554	5.2718	7.4272	13.1445
36.5406	2.2098	5.5033	7.7131	13.6640
38.0252	2.2651	5.7316	7.9968	14.1785
39.4975	2.3214	5.9571	8.2786	14.6888
40.9591	2.3788	6.1801	8.5589	15.1954
42.4119	2.4374	6.4007	8.8381	15.6988
43.8573	2.4973	6.6193	9.1166	16.1998
45.2968	2.5586	6.8359	9.3945	16.6987
46.7317	2.6214	7.0509	9.6723	17.1960
48.1632	2.6859	7.2642	9.9501	17.6921
49.5925	2.7520	7.4762	10.2282	18.1875
51.0207	2.8200	7.6869	10.5069	18.6824
52.4488	2.8900	7.8964	10.7864	19.1774
53.8779	2.9620	8.1049	11.0670	19.6726
55.3089	3.0362	8.3125	11.3488	20.1686
56.7427	3.1127	8.5193	11.6320	20.6655
58.1804	3.1917	8.7253	11.9170	21.1638
59.6228	3.2732	8.9307	12.2039	21.6637

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
61.0707	3.3574	9.1355	12.4929	22.1655
62.5251	3.4445	9.3398	12.7843	22.6696
63.9868	3.5347	9.5437	13.0783	23.1761
65.4567	3.6279	9.7472	13.3751	23.6856
66.9357	3.7246	9.9503	13.6749	24.1981
68.4245	3.8248	10.1532	13.9780	24.7141
69.9242	3.9287	10.3559	14.2846	25.2339
71.4355	4.0366	10.5584	14.5950	25.7577
72.9593	4.1486	10.7607	14.9093	26.2858
74.4967	4.2650	10.9630	15.2279	26.8186
76.0484	4.3860	11.1651	15.5511	27.3564
77.6154	4.5120	11.3671	15.8791	27.8994
79.1987	4.6431	11.5691	16.2122	28.4482
80.7994	4.7798	11.7710	16.5508	29.0029
82.4183	4.9222	11.9729	16.8951	29.5640
84.0566	5.0708	12.1747	17.2455	30.1318
85.7154	5.2260	12.3764	17.6023	30.7067
87.3959	5.3881	12.5780	17.9660	31.2891
89.0991	5.5576	12.7794	18.3370	31.8794
90.8264	5.7349	12.9807	18.7156	32.4780
92.5790	5.9206	13.1817	19.1023	33.0854
94.3584	6.1152	13.3824	19.4975	33.7021
96.1659	6.3181	13.5825	19.9006	34.3286
98.0055	6.4796	13.7746	20.2542	34.9661
99.9896	4.0490	13.6201	17.6691	35.6538
Revenue (discounted)			790.1920	
Investment cost (discounted))			67.8103	
Total profit (discounted)			722.3817	

Table 4.7 – Impulse Control solutions for $\gamma = 0$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
4.1462	1.4682	0.0000	1.4682	2.4370
7.4147	1.7204	0.7637	2.1022	3.5697
10.1649	2.0101	1.2128	2.6165	4.5229
12.6433	2.2785	1.5938	3.0754	5.3818
14.9499	2.5312	1.9389	3.5007	6.1812
17.1370	2.7731	2.2604	3.9033	6.9392
19.2361	3.0070	2.5651	4.2896	7.6667
21.2682	3.2353	2.8570	4.6638	8.3710
23.2479	3.4594	3.1390	5.0289	9.0571
25.1861	3.6805	3.4129	5.3869	9.7288
27.0909	3.8994	3.6803	5.7395	10.3890
28.9689	4.1168	3.9424	6.0880	11.0399
30.8252	4.3333	4.1999	6.4332	11.6832
32.6640	4.5493	4.4536	6.7761	12.3205
34.4889	4.7652	4.7041	7.1173	12.9529
36.3027	4.9814	4.9518	7.4574	13.5816
38.1081	5.1982	5.1972	7.7968	14.2073
39.9072	5.4157	5.4406	8.1360	14.8308
41.7019	5.6343	5.6823	8.4754	15.4528
43.4940	5.8541	5.9225	8.8153	16.0739
45.2849	6.0753	6.1615	9.1561	16.6945
47.0759	6.2982	6.3994	9.4979	17.3153
48.8684	6.5229	6.6364	9.8411	17.9365
50.6635	6.7495	6.8727	10.1859	18.5586
52.4621	6.9783	7.1083	10.5325	19.1820
54.2654	7.2094	7.3434	10.8812	19.8070
56.0743	7.4431	7.5781	11.2321	20.4339
57.8895	7.6793	7.8125	11.5856	21.0630
59.7121	7.9184	8.0466	11.9417	21.6946
61.5428	8.1606	8.2805	12.3008	22.3291
63.3824	8.4059	8.5142	12.6630	22.9667
65.2318	8.6546	8.7479	13.0286	23.6076
67.0917	8.9070	8.9815	13.3977	24.2522
68.9629	9.1631	9.2151	13.7707	24.9007

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
70.8463	9.4233	9.4486	14.1477	25.5535
72.7426	9.6878	9.6822	14.5290	26.2107
74.6526	9.9569	9.9159	14.9148	26.8726
76.5773	10.2307	10.1495	15.3055	27.5397
78.5174	10.5097	10.3832	15.7013	28.2120
80.4738	10.7940	10.6169	16.1025	28.8901
82.4476	11.0841	10.8506	16.5094	29.5741
84.4396	11.3803	11.0843	16.9224	30.2645
86.4508	11.6829	11.3179	17.3419	30.9616
88.4824	11.9925	11.5514	17.7682	31.6657
90.5354	12.3093	11.7848	18.2017	32.3772
92.6110	12.6340	12.0179	18.6429	33.0965
94.7105	12.9668	12.2506	19.0921	33.8241
96.8355	13.2846	12.4817	19.5255	34.5606
99.0358	10.8535	12.5743	17.1406	35.3232
Revenue (discounted)			802.4809	
Investment cost (discounted)			35.3109	
Total profit (discounted)			767.1700	

Table 4.8 – Impulse Control solutions for $\gamma = 0.5$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
3.8509	1.3689	0	1.3689	2.3346
7.1308	1.9589	0.7104	1.9589	3.4713
9.9511	2.4614	1.1144	2.4614	4.4488
12.5559	2.9262	1.4619	2.9262	5.3516
15.0389	3.3716	1.7809	3.3716	6.2121
17.4476	3.8067	2.0827	3.8067	7.0469
19.8100	4.2365	2.3733	4.2365	7.8656
22.1437	4.6639	2.6564	4.6639	8.6744
24.4606	5.0910	2.9343	5.0910	9.4774
26.7688	5.5191	3.2086	5.5191	10.2774
29.0742	5.9490	3.4804	5.9490	11.0763
31.3809	6.3813	3.7505	6.3813	11.8758
33.6920	6.8164	4.0195	6.8164	12.6767
36.0096	7.2545	4.2879	7.2545	13.4800
38.3355	7.6957	4.5560	7.6957	14.2861
40.6707	8.1403	4.8241	8.1403	15.0954
43.0162	8.5881	5.0924	8.5881	15.9083
45.3723	9.0393	5.3610	9.0393	16.7248
47.7396	9.4937	5.6301	9.4937	17.5453
50.1182	9.9514	5.8997	9.9514	18.3697
52.5083	10.4123	6.1700	10.4123	19.1980
54.9099	10.8763	6.4409	10.8763	20.0303
57.3230	11.3435	6.7125	11.3435	20.8666
59.7474	11.8136	6.9849	11.8136	21.7069
62.1832	12.2867	7.2580	12.2867	22.5510
64.6300	12.7627	7.5319	12.7627	23.3991
67.0879	13.2415	7.8065	13.2415	24.2509
69.5566	13.7231	8.0818	13.7231	25.1065
72.0359	14.2075	8.3579	14.2075	25.9658
74.5258	14.6945	8.6348	14.6945	26.8287
77.0260	15.1843	8.9123	15.1843	27.6952
79.5364	15.6766	9.1906	15.6766	28.5652
82.0568	16.1716	9.4696	16.1716	29.4387
84.5872	16.6692	9.7492	16.6692	30.3157

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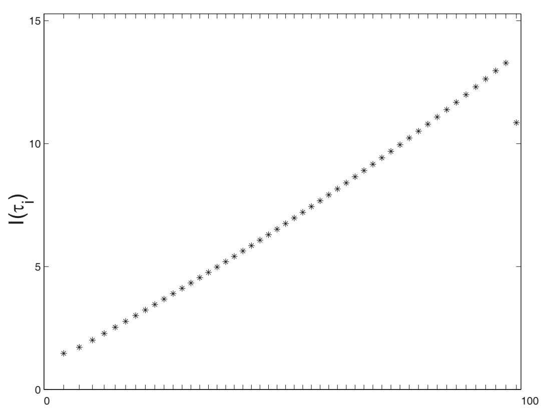
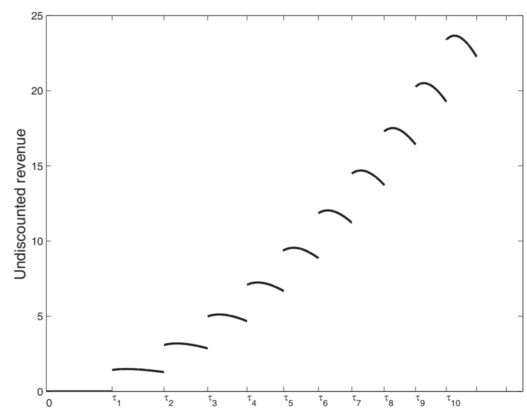
τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
87.1274	17.1693	10.0294	17.1693	31.1960
89.6772	17.6720	10.3103	17.6720	32.0798
92.2367	18.1774	10.5918	18.1774	32.9668
94.8058	18.6853	10.8739	18.6853	33.8572
97.3844	19.1959	11.1565	19.1959	34.7509
99.9725	17.0969	11.4396	17.0969	35.6478
Revenue (discounted)			771.3955	
Investment cost (discounted)			97.6050	
Total profit (discounted)			673.7904	

Table 4.9 – Impulse Control solutions for $\gamma = 1$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.



(a) Lumpy investments, $I(\tau_i)$.



(b) Undiscounted revenue for the first ten investments.

Figure 4.4 – Lumpy investments and undiscounted revenue, where $T = 100$, $r = 0.04$, $\delta = 0.05$, $\gamma = 0.5$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

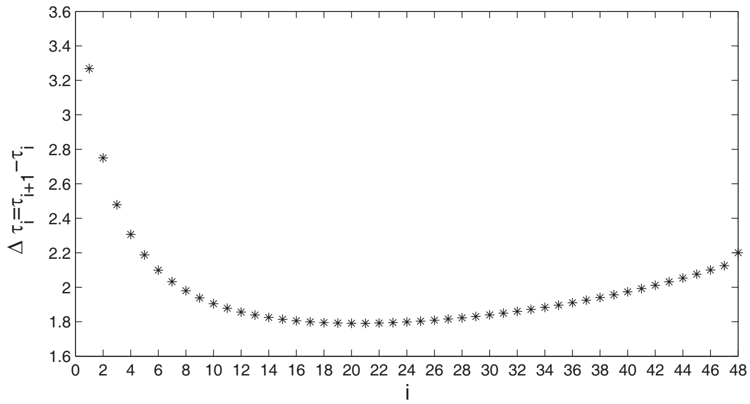
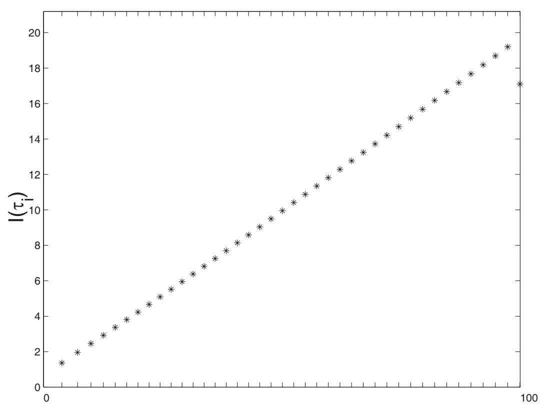
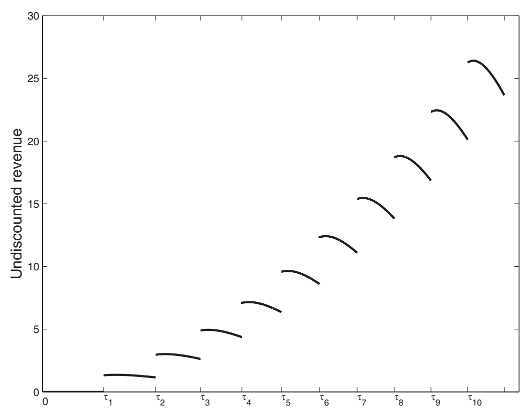


Figure 4.5 – The length between two investments for $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.



(a) Lumpy investments, $I(\tau_i)$.



(b) Undiscounted revenue for the first ten investments.

Figure 4.6 – Lumpy investments and undiscounted revenue, where $T = 100$, $r = 0.04$, $\delta = 0.05$, $\gamma = 1$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

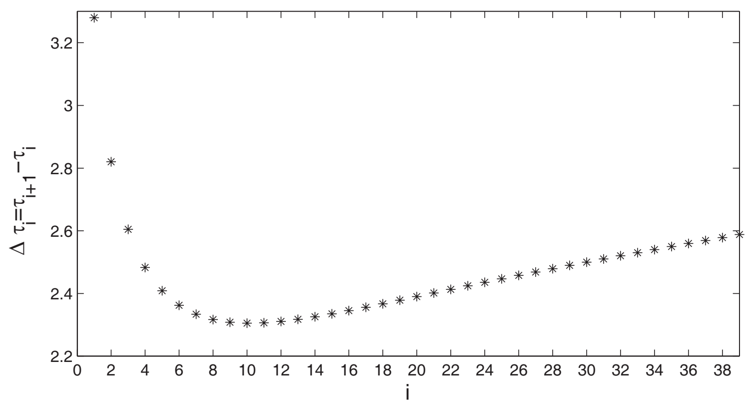


Figure 4.7 – The length between two investments for $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $\gamma = 1$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

	$b = \frac{1}{3}\log 2$	$b = \frac{1}{4}\log 2$	$b = \frac{1}{5}\log 2$	$b = \frac{1}{6}\log 2$	$b = \frac{1}{10}\log 2$
$(\tau_i : I)$	4.6759 : 1.3116	5.1658 : 1.2381	5.5832 : 1.1914	0 : 0.7418	0 : 0.7752
	8.6561 : 1.5392	9.7814 : 1.4539	10.7534 : 1.3980	7.7656 : 1.2080	9.7219 : 1.1432
	1.9662 : 1.7807	13.5911 : 1.6692	14.9977 : 1.5949	13.0448 : 1.4152	16.3705 : 1.3132
	14.9229 : 1.9995	16.9755 : 1.8614	18.7535 : 1.7685	17.4932 : 1.5907	21.9534 : 1.4524
	17.6530 : 2.2025	20.0857 : 2.0378	22.1932 : 1.9266	21.4683 : 1.7459	26.9204 : 1.5730
	20.2231 : 2.3943	23.0005 : 2.2031	25.4066 : 2.0736	25.1251 : 1.8873	31.4676 : 1.6810
	22.6732 : 2.5779	25.7678 : 2.3601	28.4478 : 2.2125	28.5485 : 2.0188	35.7025 : 1.7797
	25.0300 : 2.7553	28.4191 : 2.5108	31.3530 : 2.3450	31.7914 : 2.1429	39.6921 : 1.8712
	27.3121 : 2.9280	30.9766 : 2.6566	34.1472 : 2.4725	34.8894 : 2.2610	43.4813 : 1.9569
	29.5335 : 3.0970	33.4569 : 2.7986	36.8494 : 2.5960	37.8681 : 2.3745	47.1023 : 2.0376
	31.7048 : 3.2631	35.8726 : 2.9373	39.4737 : 2.7162	40.7466 : 2.4841	50.5789 : 2.1141
	33.8341 : 3.4270	38.2335 : 3.0736	42.0316 : 2.8337	43.5397 : 2.5905	53.9292 : 2.1869
	35.9284 : 3.5893	40.5479 : 3.2079	44.5320 : 2.9489	46.2589 : 2.6942	57.1675 : 2.2563
	37.9929 : 3.7502	42.8221 : 3.3405	46.9826 : 3.0623	48.9138 : 2.7957	60.3051 : 2.3226
	40.0324 : 3.9103	45.0617 : 3.4720	49.3895 : 3.1741	51.5122 : 2.8952	63.3512 : 2.3861
	42.0508 : 4.0697	47.2715 : 3.6024	51.7580 : 3.2847	54.0606 : 2.9931	66.3131 : 2.4470
	44.0513 : 4.2289	49.4554 : 3.7322	54.0927 : 3.3943	56.5646 : 3.0897	69.1972 : 2.5052
	46.0368 : 4.3879	51.6169 : 3.8615	56.3977 : 3.5031	59.0290 : 3.1851	72.0083 : 2.5610
	48.0100 : 4.5471	53.7592 : 3.9905	58.6762 : 3.6113	61.4580 : 3.2795	74.7508 : 2.6143
	49.9730 : 4.7067	55.8849 : 4.1195	60.9316 : 3.7190	63.8553 : 3.3731	77.4280 : 2.6652
	51.9280 : 4.8668	57.9966 : 4.2485	63.1664 : 3.8265	66.2241 : 3.4661	80.0427 : 2.7137
	53.8767 : 5.0275	60.0964 : 4.3779	65.3833 : 3.9339	68.5673 : 3.5586	82.5974 : 2.7599
	55.8207 : 5.1892	62.1864 : 4.5077	67.5845 : 4.0413	70.8876 : 3.6507	85.0937 : 2.8035
	57.7618 : 5.3518	64.2685 : 4.6381	69.7722 : 4.1488	73.1873 : 3.7426	87.5331 : 2.8447
	59.7012 : 5.5156	66.3445 : 4.7692	71.9481 : 4.2566	75.4687 : 3.8343	89.9166 : 2.8834
	61.6403 : 5.6808	68.4159 : 4.9012	74.1143 : 4.3649	77.7338 : 3.9260	92.2448 : 2.9194

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	$b=\frac{1}{3}\log 2$	$b=\frac{1}{4}\log 2$	$b=\frac{1}{5}\log 2$	$b=\frac{1}{6}\log 2$	$b=\frac{1}{10}\log 2$
$(\tau_i : I)$	63.5803 : 5.8475	70.4843 : 5.0342	76.2723 : 4.4736	79.9845 : 4.0178	94.5183 : 2.9527
	65.5226 : 6.0158	72.5512 : 5.1684	78.4238 : 4.5831	82.2225 : 4.1098	96.7374 : 2.9784
	67.4682 : 6.1859	74.6181 : 5.3039	80.5704 : 4.6933	84.4495 : 4.2021	98.9488 : 2.4142
	69.4182 : 6.3579	76.6862 : 5.4409	82.7134 : 4.8044	86.6672 : 4.2948	
	71.3738 : 6.5321	78.7570 : 5.5795	84.8544 : 4.9166	88.8770 : 4.3879	
	73.3360 : 6.7086	80.8317 : 5.7199	86.9948 : 5.0299	91.0803 : 4.4817	
	75.3058 : 6.8876	82.9116 : 5.8623	89.1359 : 5.1445	93.2787 : 4.5762	
	77.2844 : 7.0692	84.9980 : 6.0068	91.2791 : 5.2606	95.4734 : 4.6715	
	79.2727 : 7.2536	87.0922 : 6.1536	93.4257 : 5.3783	97.6662 : 4.7607	
	81.2718 : 7.4412	89.1954 : 6.3029	95.5770 : 5.4977	99.9020 : 3.9729	
	83.2828 : 7.6320	91.3090 : 6.4549	97.7349 : 5.6100		
	85.3067 : 7.8263	93.4344 : 6.6098	99.9462 : 4.6388		
	87.3447 : 8.0243	95.5727 : 6.7678			
	89.3980 : 8.2264	97.7260 : 6.9181			
	91.4676 : 8.4327	99.9410 : 5.7311			
	93.5549 : 8.6437				
	95.6610 : 8.8594				
	97.7878 : 9.0647				
	99.9841 : 7.4363				
Rev	371.5616	220.0775	148.0959	108.6965	47.4170
ICost	39.2258	27.6829	21.7123	19.5772	12.9673
Profit	332.3358	192.3946	126.3837	89.1193	34.4497

Table 4.10 – Impulse Control solutions for different b, where $T = 100$, $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
4.6759	1.3116	0.0000	1.3116	2.0804
8.6561	1.5392	0.5917	1.8351	3.0000
11.9662	1.7807	0.9465	2.2540	3.7648
14.9229	1.9995	1.2478	2.6234	4.4479
17.6530	2.2025	1.5196	2.9623	5.0787
20.2231	2.3943	1.7717	3.2801	5.6725
22.6732	2.5779	2.0094	3.5826	6.2386
25.0300	2.7553	2.2361	3.8733	6.7832
27.3121	2.9280	2.4539	4.1549	7.3104
29.5335	3.0970	2.6645	4.4292	7.8237

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
31.7048	3.2631	2.8690	4.6976	8.3254
33.8341	3.4270	3.0685	4.9613	8.8173
35.9284	3.5893	3.2636	5.2210	9.3012
37.9929	3.7502	3.4549	5.4776	9.7782
40.0324	3.9103	3.6429	5.7317	10.2495
42.0508	4.0697	3.8280	5.9837	10.7158
44.0513	4.2289	4.0106	6.2342	11.1780
46.0368	4.3879	4.1910	6.4834	11.6368
48.0100	4.5471	4.3693	6.7318	12.0927
49.9730	4.7067	4.5459	6.9797	12.5462
51.9280	4.8668	4.7209	7.2272	12.9979
53.8767	5.0275	4.8945	7.4748	13.4482
55.8207	5.1892	5.0669	7.7226	13.8973
57.7618	5.3518	5.2380	7.9708	14.3458
59.7012	5.5156	5.4082	8.2197	14.7939
61.6403	5.6808	5.5774	8.4695	15.2419
63.5803	5.8475	5.7457	8.7203	15.6902
65.5226	6.0158	5.9133	8.9724	16.1389
67.4682	6.1859	6.0802	9.2260	16.5885
69.4182	6.3579	6.2465	9.4812	17.0390
71.3738	6.5321	6.4121	9.7382	17.4909
73.3360	6.7086	6.5773	9.9972	17.9442
75.3058	6.8876	6.7419	10.2585	18.3993
77.2844	7.0692	6.9061	10.5222	18.8565
79.2727	7.2536	7.0698	10.7885	19.3159
81.2718	7.4412	7.2331	11.0577	19.7778
83.2828	7.6320	7.3959	11.3299	20.2424
85.3067	7.8263	7.5584	11.6055	20.7100
87.3447	8.0243	7.7204	11.8845	21.1809
89.398	8.2264	7.8821	12.1674	21.6553
91.4676	8.4327	8.0433	12.4544	22.1335
93.5549	8.6437	8.204	12.7457	22.6158
95.661	8.8594	8.3642	13.0415	23.1024
97.7878	9.0647	8.5231	13.3263	23.5938
99.9841	7.4363	8.5889	11.7308	24.1012
Revenue (discounted)			371.5616	
Investment cost (discounted)			39.2258	
Total profit (discounted)			332.3358	

Table 4.11 – Impulse Control solutions for $b = \frac{1}{3}\log 2$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
5.1658	1.2381	0	1.2381	1.8952
9.7814	1.4539	0.4919	1.6998	2.695
13.5911	1.6692	0.7934	2.0659	3.3552
16.9755	1.8614	1.0499	2.3864	3.9416
20.0857	2.0378	1.2811	2.6784	4.4806
23.0005	2.2031	1.4952	2.9507	4.9857
25.7678	2.3601	1.6965	3.2084	5.4652
28.4191	2.5108	1.888	3.4548	5.9246
30.9766	2.6566	2.0715	3.6924	6.3678
33.4569	2.7986	2.2484	3.9228	6.7976
35.8726	2.9373	2.4197	4.1472	7.2162
38.2335	3.0736	2.5863	4.3668	7.6254
40.5479	3.2079	2.7488	4.5823	8.0264
42.8221	3.3405	2.9077	4.7944	8.4205
45.0617	3.472	3.0634	5.0036	8.8086
47.2715	3.6024	3.2162	5.2105	9.1915
49.4554	3.7322	3.3666	5.4155	9.5700
51.6169	3.8615	3.5147	5.6188	9.9445
53.7592	3.9905	3.6607	5.8209	10.3158
55.8849	4.1195	3.805	6.0219	10.6841
57.9966	4.2485	3.9475	6.2223	11.0500
60.0964	4.3779	4.0885	6.4221	11.4139
62.1864	4.5077	4.2281	6.6217	11.7761
64.2685	4.6381	4.3664	6.8213	12.1369
66.3445	4.7692	4.5035	7.0210	12.4966
68.4159	4.9012	4.6395	7.2210	12.8556
70.4843	5.0342	4.7746	7.4215	13.214
72.5512	5.1684	4.9086	7.6227	13.5722
74.6181	5.3039	5.0418	7.8248	13.9303
76.6862	5.4409	5.1741	8.0280	14.2887
78.7570	5.5795	5.3057	8.2324	14.6475
80.8317	5.7199	5.4365	8.4382	15.0071

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
82.9116	5.8623	5.5666	8.6456	15.3675
84.9980	6.0068	5.6961	8.8548	15.729
87.0922	6.1536	5.8248	9.0660	16.0919
89.1954	6.3029	5.9529	9.2794	16.4564
91.3090	6.4549	6.0804	9.4951	16.8227
93.4344	6.6098	6.2072	9.7134	17.1909
95.5727	6.7678	6.3334	9.9345	17.5615
97.7260	6.9181	6.4583	10.1472	17.9346
99.9410	5.7311	6.5156	8.9889	18.3185
Revenue (discounted)			220.0775	
Investment cost (discounted)			27.6829	
Total profit (discounted)			192.3946	

Table 4.12 – Impulse Control solutions for $b = \frac{1}{4}\log 2$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
5.5832	1.1914	0	1.1914	1.774
10.7534	1.398	0.4236	1.6098	2.4907
14.9977	1.5949	0.6888	1.9393	3.0791
18.7535	1.7685	0.915	2.226	3.5998
22.1932	1.9266	1.1188	2.486	4.0766
25.4066	2.0736	1.3073	2.7273	4.5221
28.4478	2.2125	1.4845	2.9547	4.9437
31.353	2.345	1.6527	3.1714	5.3464
34.1472	2.4725	1.8136	3.3793	5.7338
36.8494	2.596	1.9685	3.5802	6.1084
39.4737	2.7162	2.1182	3.7753	6.4722
42.0316	2.8337	2.2635	3.9654	6.8268
44.532	2.9489	2.4049	4.1514	7.1734
46.9826	3.0623	2.543	4.3338	7.5132
49.3895	3.1741	2.678	4.5131	7.8468
51.758	3.2847	2.8103	4.6899	8.1752
54.0927	3.3943	2.9401	4.8644	8.4988
56.3977	3.5031	3.0678	5.037	8.8184
58.6762	3.6113	3.1934	5.208	9.1343
60.9316	3.719	3.3172	5.3777	9.4469

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
63.1664	3.8265	3.4393	5.5462	9.7567
65.3833	3.9339	3.5599	5.7138	10.0641
67.5845	4.0413	3.679	5.8808	10.3692
69.7722	4.1488	3.7968	6.0472	10.6725
71.9481	4.2566	3.9134	6.2133	10.9741
74.1143	4.3649	4.0288	6.3793	11.2744
76.2723	4.4736	4.1431	6.5452	11.5736
78.4238	4.5831	4.2564	6.7113	11.8718
80.5704	4.6933	4.3688	6.8776	12.1694
82.7134	4.8044	4.4802	7.0445	12.4665
84.8544	4.9166	4.5907	7.2119	12.7633
86.9948	5.0299	4.7005	7.3801	13.0600
89.1359	5.1445	4.8094	7.5492	13.3569
91.2791	5.2606	4.9176	7.7194	13.6540
93.4257	5.3783	5.0250	7.8908	13.9515
95.5770	5.4977	5.1317	8.0635	14.2498
97.7349	5.6100	5.2372	8.2286	14.5489
99.9462	4.6388	5.2876	7.2826	14.8555
Revenue (discounted)			148.0959	
Investment cost (discounted)			21.7123	
Total profit (discounted)			126.3837	

Table 4.13 – Impulse Control solutions for $b = \frac{1}{5}\log 2$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
0	0.7418	0	0.7418	1
7.7656	1.208	0.1570	1.2865	1.8971
13.0448	1.4152	0.4476	1.6390	2.5070
17.4932	1.5907	0.6733	1.9274	3.0209
21.4683	1.7459	0.8703	2.1811	3.4801
25.1251	1.8873	1.0496	2.4121	3.9026
28.5485	2.0188	1.2163	2.6270	4.2981
31.7914	2.1429	1.3734	2.8296	4.6727

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
34.8894	2.2610	1.5227	3.0224	5.0306
37.8681	2.3745	1.6658	3.2074	5.3747
40.7466	2.4841	1.8036	3.3859	5.7072
43.5397	2.5905	1.9367	3.5589	6.0299
46.2589	2.6942	2.0660	3.7272	6.3440
48.9138	2.7957	2.1917	3.8915	6.6507
51.5122	2.8952	2.3143	4.0524	6.9509
54.0606	2.9931	2.4342	4.2102	7.2453
56.5646	3.0897	2.5516	4.3655	7.5346
59.0290	3.1851	2.6667	4.5184	7.8193
61.4580	3.2795	2.7798	4.6694	8.0999
63.8553	3.3731	2.8909	4.8186	8.3769
66.2241	3.4661	3.0003	4.9663	8.6505
68.5673	3.5586	3.1082	5.1127	8.9212
70.8876	3.6507	3.2145	5.2580	9.1893
73.1873	3.7426	3.3194	5.4023	9.4549
75.4687	3.8343	3.4231	5.5459	9.7185
77.7338	3.9260	3.5255	5.6888	9.9802
79.9845	4.0178	3.6268	5.8312	10.2402
82.2225	4.1098	3.7271	5.9733	10.4987
84.4495	4.2021	3.8263	6.1152	10.7560
86.6672	4.2948	3.9245	6.2570	11.0122
88.8770	4.3879	4.0219	6.3989	11.2675
91.0803	4.4817	4.1183	6.5409	11.5220
93.2787	4.5762	4.2140	6.6832	11.7760
95.4734	4.6715	4.3088	6.8259	12.0295
97.6662	4.7607	4.4024	6.9619	12.2828
99.9020	3.9729	4.4517	6.1988	12.5411
Revenue (discounted)			108.6965	
Investment cost (discounted)			19.5772	
Total profit (discounted)			59.1193	

Table 4.14 – Impulse Control solutions for $b = \frac{1}{6}\log 2$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
0	0.7752	0	0.7752	1
9.7219	1.1432	0.1109	1.1987	1.6739
16.3705	1.3132	0.3171	1.4717	2.1347
21.9534	1.4524	0.4818	1.6934	2.5217
26.9204	1.5730	0.6271	1.8866	2.8660
31.4676	1.6810	0.7598	2.0609	3.1812
35.7025	1.7797	0.8835	2.2215	3.4747
39.6921	1.8712	1.0003	2.3713	3.7512
43.4813	1.9569	1.1114	2.5125	4.0139
47.1023	2.0376	1.2179	2.6465	4.2649
50.5789	2.1141	1.3204	2.7743	4.5059
53.9292	2.1869	1.4195	2.8966	4.7381
57.1675	2.2563	1.5157	3.0142	4.9626
60.3051	2.3226	1.6093	3.1273	5.1800
63.3512	2.3861	1.7006	3.2364	5.3912
66.3131	2.4470	1.7897	3.3418	5.5965
69.1972	2.5052	1.8771	3.4437	5.7964
72.0083	2.5610	1.9627	3.5423	5.9912
74.7508	2.6143	2.0468	3.6377	6.1813
77.4280	2.6652	2.1295	3.7300	6.3669
80.0427	2.7137	2.2110	3.8192	6.5481
82.5974	2.7599	2.2913	3.9055	6.7252
85.0937	2.8035	2.3706	3.9888	6.8982
87.5331	2.8447	2.4489	4.0692	7.0673
89.9166	2.8834	2.5263	4.1465	7.2325
92.2448	2.9194	2.6029	4.2208	7.3939
94.5183	2.9527	2.6787	4.2920	7.5515
96.7374	2.9784	2.7537	4.3553	7.7053
98.9488	2.4142	2.7985	3.8134	7.8586
Revenue (discounted)			47.417	
Investment cost (discounted)			12.9673	
Total profit (discounted)			34.4497	

Table 4.15 – Impulse Control solutions for $b = \frac{1}{10}\log 2$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

	$C = 4$	$C = 8$	$C = 16$	$C = 32$
$(\tau_i : I)$	5.7915 : 1.8832	8.0844 : 2.4856	11.1517 : 3.3199	15.2866 : 4.4754
	9.6593 : 2.2099	12.7147 : 2.9206	16.6712 : 3.8947	21.8148 : 5.2241
	12.8816 : 2.5607	16.5386 : 3.3546	21.1933 : 4.4297	27.1293 : 5.8789
	15.7638 : 2.8797	19.9372 : 3.7422	25.1901 : 4.8993	31.8052 : 6.4443
	18.4283 : 3.1763	23.0621 : 4.0984	28.8471 : 5.3256	36.0657 : 6.9513
	20.9394 : 3.4571	25.9923 : 4.4325	32.2606 : 5.7215	40.0266 : 7.4169
	23.3358 : 3.7265	28.7755 : 4.7502	35.4889 : 6.0947	43.7577 : 7.8515
	25.6433 : 3.9871	31.4435 : 5.0556	38.5705 : 6.4506	47.3050 : 8.2618
	27.8799 : 4.2412	34.0186 : 5.3513	41.5327 : 6.7926	50.7012 : 8.6525
	30.0590 : 4.4903	36.5173 : 5.6394	44.3957 : 7.1237	53.9701 : 9.0270
	32.1907 : 4.7354	38.9523 : 5.9215	47.1748 : 7.4458	57.1301 : 9.3879
	34.2832 : 4.9775	41.3336 : 6.1989	49.8823 : 7.7606	60.1956 : 9.7373
	36.3429 : 5.2174	43.6692 : 6.4726	52.5280 : 8.0694	63.1782 : 10.0767
	38.3751 : 5.4557	45.9658 : 6.7433	55.1200 : 8.3732	66.0873 : 10.4075
	40.3842 : 5.6929	48.2290 : 7.0118	57.6651 : 8.6731	68.9309 : 10.7307
	42.3740 : 5.9295	50.4635 : 7.2788	60.1692 : 8.9698	71.7156 : 11.0472
	44.3476 : 6.1658	52.6734 : 7.5447	62.6371 : 9.2641	74.4470 : 11.3579
	46.3080 : 6.4022	54.8622 : 7.8101	65.0733 : 9.5565	77.1303 : 11.6634
	48.2575 : 6.6390	57.0331 : 8.0754	67.4816 : 9.8476	79.7695 : 11.9643
	50.1983 : 6.8765	59.1889 : 8.3409	69.8655 : 10.1379	82.3685 : 12.2611
	52.1324 : 7.1149	61.3321 : 8.6072	72.2281 : 10.4280	84.9306 : 12.5542
	54.0615 : 7.3546	63.4651 : 8.8745	74.5721 : 10.7183	87.4588 : 12.8441
	55.9872 : 7.5957	65.5899 : 9.1432	76.9002 : 11.0092	89.9557 : 13.1311
	57.9112 : 7.8386	67.7086 : 9.4136	79.2148 : 11.3012	92.4238 : 13.4156
	59.8346 : 8.0833	69.8230 : 9.6860	81.5182 : 11.5947	94.8653 : 13.6978
	61.7589 : 8.3302	71.9347 : 9.9609	83.8124 : 11.8901	97.2825 : 13.9625
	63.6852 : 8.5795	74.0455 : 10.2384	86.0995 : 12.1878	99.7136 : 12.0324
	65.6147 : 8.8313	76.1570 : 10.5191	88.3813 : 12.4883	
	67.5486 : 9.0860	78.2706 : 10.8031	90.6599 : 12.7920	
	69.4879 : 9.3438	80.3880 : 11.0909	92.9369 : 13.0994	
	71.4338 : 9.6049	82.5104 : 11.3829	95.2143 : 13.4107	
	73.3871 : 9.8696	84.6394 : 11.6793	97.4939 : 13.7081	
	75.3490 : 10.1381	86.7765 : 11.9807	99.8182 : 11.6236	

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	$C=4$	$C=8$	$C=16$	$C=32$
$(\tau_i : I)$	77.3205 : 10.4107	88.9230 : 12.2875		
	79.3027 : 10.6877	91.0805 : 12.6000		
	81.2965 : 10.9694	93.2503 : 12.9188		
	83.3031 : 11.2562	95.4341 : 13.2441		
	85.3235 : 11.5483	97.6337 : 13.5562		
	87.3588 : 11.8462	99.8944 : 11.3351		
	89.4101 : 12.1503			
	91.4786 : 12.4609			
	93.5657 : 12.7786			
	95.6724 : 13.1036			
	97.8006 : 13.4128			
Rev	780.7835	769.1875	747.0746	712.6433
ICost	79.5936	96.8939	120.5584	150.9987
Profit	701.1899	672.2936	626.5162	561.6447

Table 4.16 – Impulse Control solutions for different C, where $T = 100$, $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
5.7915	1.8832	0	1.8832	3.0072
9.6593	2.2099	0.8688	2.6444	4.3477
12.8816	2.5607	1.3881	3.2548	5.4644
15.7638	2.8797	1.8289	3.7941	6.4633
18.4283	3.1763	2.2267	4.2897	7.3868
20.9394	3.4571	2.5960	4.7552	8.2570
23.3358	3.7265	2.9445	5.1987	9.0876
25.6433	3.9871	3.2770	5.6256	9.8873
27.8799	4.2412	3.5967	6.0396	10.6624
30.0590	4.4903	3.9060	6.4433	11.4176
32.1907	4.7354	4.2067	6.8387	12.1565
34.2832	4.9775	4.5001	7.2276	12.8817
36.3429	5.2174	4.7873	7.6111	13.5955

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
38.3751	5.4557	5.0691	7.9903	14.2998
40.3842	5.6929	5.3463	8.3661	14.9961
42.3740	5.9295	5.6194	8.7392	15.6857
44.3476	6.1658	5.8890	9.1103	16.3697
46.3080	6.4022	6.1554	9.4799	17.0491
48.2575	6.6390	6.4191	9.8485	17.7248
50.1983	6.8765	6.6803	10.2166	18.3974
52.1324	7.1149	6.9393	10.5846	19.0677
54.0615	7.3546	7.1963	10.9528	19.7363
55.9872	7.5957	7.4517	11.3216	20.4037
57.9112	7.8386	7.7054	11.6913	21.0705
59.8346	8.0833	7.9578	12.0622	21.7371
61.7589	8.3302	8.2089	12.4347	22.4040
63.6852	8.5795	8.4589	12.8089	23.0716
65.6147	8.8313	8.7079	13.1853	23.7403
67.5486	9.0860	8.9560	13.5640	24.4106
69.4879	9.3438	9.2033	13.9455	25.0827
71.4338	9.6049	9.4498	14.3298	25.7571
73.3871	9.8696	9.6956	14.7174	26.4340
75.3490	10.1381	9.9408	15.1085	27.1140
77.3205	10.4107	10.1853	15.5034	27.7973
79.3027	10.6877	10.4294	15.9024	28.4842
81.2965	10.9694	10.6728	16.3058	29.1752
83.3031	11.2562	10.9158	16.7141	29.8707
85.3235	11.5483	11.1582	17.1275	30.5709
87.3588	11.8462	11.4001	17.5463	31.2762
89.4101	12.1503	11.6415	17.9711	31.9872
91.4786	12.4609	11.8823	18.4021	32.7041
93.5657	12.7786	12.1225	18.8398	33.4274
95.6724	13.1036	12.3620	19.2846	34.1575
97.8006	13.4128	12.5995	19.7126	34.8951
Revenue (discounted)			780.7835	
Investment cost (discounted)			79.5936	
Total profit (discounted)			701.1899	

Table 4.17 – Impulse Control solutions for $C = 4$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
8.0844	2.4856	0.0000	2.4856	3.8018
12.7147	2.9206	0.9846	3.4129	5.4066
16.5386	3.3546	1.5885	4.1489	6.7318
19.9372	3.7422	2.1025	4.7935	7.9097
23.0621	4.0984	2.5658	5.3813	8.9927
25.9923	4.4325	2.9949	5.9299	10.0082
28.7755	4.7502	3.3986	6.4495	10.9728
31.4435	5.0556	3.7826	6.9468	11.8975
34.0186	5.3513	4.1507	7.4266	12.7899
36.5173	5.6394	4.5056	7.8922	13.6559
38.9523	5.9215	4.8495	8.3463	14.4998
41.3336	6.1989	5.1839	8.7909	15.3251
43.6692	6.4726	5.5101	9.2276	16.1346
45.9658	6.7433	5.8292	9.6579	16.9305
48.2290	7.0118	6.1419	10.0828	17.7149
50.4635	7.2788	6.4491	10.5033	18.4893
52.6734	7.5447	6.7512	10.9203	19.2552
54.8622	7.8101	7.0489	11.3345	20.0138
57.0331	8.0754	7.3425	11.7466	20.7662
59.1889	8.3409	7.6324	12.1572	21.5133
61.3321	8.6072	7.9190	12.5667	22.2561
63.4651	8.8745	8.2026	12.9758	22.9953
65.5899	9.1432	8.4834	13.3849	23.7317
67.7086	9.4136	8.7617	13.7944	24.4660
69.8230	9.6860	9.0376	14.2048	25.1988
71.9347	9.9609	9.3113	14.6165	25.9307
74.0455	10.2384	9.5830	15.0299	26.6622
76.1570	10.5191	9.8527	15.4454	27.3940
78.2706	10.8031	10.1207	15.8635	28.1265
80.3880	11.0909	10.3870	16.2844	28.8603
82.5104	11.3829	10.6517	16.7087	29.5959
84.6394	11.6793	10.9149	17.1368	30.3338
86.7765	11.9807	11.1765	17.5690	31.0744
88.9230	12.2875	11.4367	18.0058	31.8184
91.0805	12.6000	11.6955	18.4477	32.5661
93.2503	12.9188	11.9528	18.8952	33.3181
95.4341	13.2441	12.2087	19.3485	34.0749
97.6337	13.5562	12.4621	19.7872	34.8373
99.8944	11.3351	12.5899	17.6300	35.6208
Revenue (discounted)			769.1875	
Investment cost (discounted)			96.8939	
Total profit (discounted)			672.2936	

Table 4.18 – Impulse Control solutions for $C = 8$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
11.1517	3.3199	0.0000	3.3199	4.8649
16.6712	3.8947	1.1008	4.4451	6.7778
21.1933	4.4297	1.7993	5.3294	8.3450
25.1901	4.8993	2.3961	6.0974	9.7302
28.8471	5.3256	2.9343	6.7928	10.9976
32.2606	5.7215	3.4320	7.4375	12.1807
35.4889	6.0947	3.8996	8.0445	13.2995
38.5705	6.4506	4.3434	8.6223	14.3675
41.5327	6.7926	4.7679	9.1766	15.3941
44.3957	7.1237	5.1762	9.7118	16.3864
47.1748	7.4458	5.5707	10.2312	17.3495
49.8823	7.7606	5.9533	10.7372	18.2879
52.5280	8.0694	6.3254	11.2321	19.2048
55.1200	8.3732	6.6884	11.7174	20.1031
57.6651	8.6731	7.0431	12.1947	20.9852
60.1692	8.9698	7.3905	12.6651	21.8530
62.6371	9.2641	7.7312	13.1297	22.7084
65.0733	9.5565	8.0658	13.5894	23.5527
67.4816	9.8476	8.3949	14.0450	24.3874
69.8655	10.1379	8.7189	14.4974	25.2135
72.2281	10.4280	9.0382	14.9471	26.0323
74.5721	10.7183	9.3531	15.3949	26.8447
76.9002	11.0092	9.6640	15.8412	27.6516
79.2148	11.3012	9.9711	16.2868	28.4538
81.5182	11.5947	10.2747	16.7321	29.2520
83.8124	11.8901	10.5749	17.1776	30.0472
86.0995	12.1878	10.8720	17.6238	30.8398
88.3813	12.4883	11.1660	18.0714	31.6306
90.6599	12.7920	11.4572	18.5206	32.4203
92.9369	13.0994	11.7456	18.9722	33.2095
95.2143	13.4107	12.0313	19.4263	33.9987
97.4939	13.7081	12.3135	19.8649	34.7888
99.8182	11.6236	12.4796	17.8634	35.5944
Revenue (discounted)			747.0746	
Investment cost (discounted)			120.5584	
Total profit (discounted)			626.5162	

Table 4.19 – Impulse Control solutions for $C = 16$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
15.2866	4.4754	0.0000	4.4754	6.2979
21.8148	5.2241	1.2128	5.8305	8.5604
27.1293	5.8789	2.0142	6.8860	10.4023
31.8052	6.4443	2.7029	7.7957	12.0228
36.0657	6.9513	3.3250	8.6138	13.4994
40.0266	7.4169	3.9008	9.3673	14.8722
43.7577	7.8515	4.4416	10.0723	16.1653
47.3050	8.2618	4.9546	10.7391	17.3947
50.7012	8.6525	5.4448	11.3748	18.5717
53.9701	9.0270	5.9158	11.9848	19.7046
57.1301	9.3879	6.3703	12.5730	20.7998
60.1956	9.7373	6.8104	13.1425	21.8622
63.1782	10.0767	7.2379	13.6956	22.8959
66.0873	10.4075	7.6542	14.2345	23.9041
68.9309	10.7307	8.0604	14.7609	24.8896
71.7156	11.0472	8.4574	15.2759	25.8547
74.4470	11.3579	8.8461	15.7810	26.8014
77.1303	11.6634	9.2273	16.2771	27.7313
79.7695	11.9643	9.6014	16.7650	28.6460
82.3685	12.2611	9.9692	17.2457	29.5467
84.9306	12.5542	10.3309	17.7196	30.4347
87.4588	12.8441	10.6871	18.1876	31.3109
89.9557	13.1311	11.0381	18.6501	32.1763
92.4238	13.4156	11.3842	19.1077	33.0317
94.8653	13.6978	11.7258	19.5607	33.8778
97.2825	13.9625	12.0624	19.9937	34.7155
99.7136	12.0324	12.2950	18.1799	35.5581
Revenue (discounted)			712.6433	
Investment cost (discounted)			150.9987	
Total profit (discounted)			561.6447	

Table 4.20 – Impulse Control solutions for $C = 32$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $K_0 = 0$ and $\theta(0) = 1$.

	$\eta = 0.01$	$\eta = 0.02$	$\eta = 0.03$
$(\tau_i : I)$	5.2730 : 1.7250	6.3504 : 1.9594	7.5126 : 2.2042
	8.9696 : 2.0366	10.4003 : 2.3175	11.902 : 2.6060
	12.0850 : 2.3821	13.8098 : 2.7062	15.5932 : 3.0359
	14.9011 : 2.7029	16.8941 : 3.0676	18.9345 : 3.4366
	17.5308 : 3.0067	19.7779 : 3.4110	22.0629 : 3.8188
	20.0327 : 3.2991	22.5261 : 3.7427	25.0493 : 4.1897
	22.4425 : 3.5837	25.1779 : 4.0670	27.9368 : 4.5542
	24.7835 : 3.8631	27.7594 : 4.3869	30.7539 : 4.9156
	27.0723 : 4.1393	30.2889 : 4.7047	33.5212 : 5.2765
	29.3212 : 4.4136	32.7803 : 5.0219	36.2541 : 5.6390
	31.5397 : 4.6871	35.2443 : 5.3400	38.9647 : 6.0049
	33.7353 : 4.9607	37.6894 : 5.6602	41.6632 : 6.3757
	35.9140 : 5.2353	40.1229 : 5.9836	44.3578 : 6.7529
	38.0810 : 5.5115	42.5509 : 6.3111	47.0561 : 7.1378
	40.2406 : 5.7900	44.9786 : 6.6436	49.7647 : 7.5319
	42.3967 : 6.0713	47.411 : 6.9821	52.4898 : 7.9365
	44.5525 : 6.3560	49.8523 : 7.3274	55.2371 : 8.3532
	46.7110 : 6.6445	52.3066 : 7.6804	58.0124 : 8.7834
	48.8751 : 6.9374	54.7778 : 8.0421	60.8212 : 9.2288
	51.0474 : 7.2352	57.2696 : 8.4135	63.6691 : 9.6913
	53.2301 : 7.5385	59.7857 : 8.7955	66.562 : 10.1729
	55.4257 : 7.8477	62.3297 : 9.1893	69.5058 : 10.6757
	57.6364 : 8.1634	64.9055 : 9.5961	72.5068 : 11.2023
	59.8643 : 8.4863	67.5167 : 10.0172	75.5718 : 11.7556
	62.1118 : 8.8168	70.1673 : 10.4539	78.7082 : 12.3386
	64.3809 : 9.1557	72.8614 : 10.9080	81.9241 : 12.9551
	66.6739 : 9.5037	75.6034 : 11.3811	85.2281 : 13.6094
	68.9931 : 9.8616	78.3978 : 11.8752	88.6304 : 14.3064
	71.3408 : 10.2301	81.2498 : 12.3925	92.1421 : 15.0518
	73.7194 : 10.6101	84.1647 : 12.9354	95.7764 : 15.8177
	76.1314 : 11.0026	87.1483 : 13.5069	99.595 : 12.3688

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	$\eta = 0.01$	$\eta = 0.02$	$\eta = 0.03$
$(\tau_i : I)$	78.5796 : 11.4087	90.2071 : 14.1101	
	81.0668 : 11.8296	93.3484 : 14.7483	
	83.5959 : 12.2664	96.5805 : 15.3926	
	86.1701 : 12.7208	99.9593 : 12.1273	
	88.793 : 13.1942		
	91.4683 : 13.6885		
	94.1999 : 14.2053		
	96.9928 : 14.7182		
	99.8973 : 11.8801		
Rev	762.5966	733.2291	701.2148
ICost	61.1145	56.6083	52.6074
Profit	701.4821	676.6208	648.6074

Table 4.21 – Impulse Control solutions for different η, where $T = 100$, $\gamma = 0.5$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$. Furthermore, Rev and ICost denote the discounted revenue and the discounted investment cost, respectively.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
5.2730	1.7250	0.0000	1.725	2.8275
8.9696	2.0366	0.8236	2.4483	4.1086
12.0850	2.3821	1.3130	3.0386	5.1884
14.9011	2.7029	1.7301	3.568	6.1643
17.5308	3.0067	2.1087	4.0611	7.0757
20.0327	3.2991	2.4622	4.5302	7.9428
22.4425	3.5837	2.7977	4.9825	8.7780
24.7835	3.8631	3.1197	5.423	9.5893
27.0723	4.1393	3.4311	5.8548	10.3825
29.3212	4.4136	3.7340	6.2806	11.1620
31.5397	4.6871	4.0300	6.702	11.9308
33.7353	4.9607	4.3202	7.1208	12.6918
35.9140	5.2353	4.6056	7.5381	13.4469
38.0810	5.5115	4.8870	7.955	14.1979
40.2406	5.7900	5.1649	8.3724	14.9463
42.3967	6.0713	5.4398	8.7912	15.6936
44.5525	6.3560	5.7121	9.212	16.4407
46.7110	6.6445	5.9822	9.6356	17.1888

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τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
48.8751	6.9374	6.2504	10.0626	17.9388
51.0474	7.2352	6.5168	10.4936	18.6917
53.2301	7.5385	6.7816	10.9293	19.4481
55.4257	7.8477	7.0451	11.3702	20.2091
57.6364	8.1634	7.3072	11.817	20.9752
59.8643	8.4863	7.5681	12.2703	21.7474
62.1118	8.8168	7.8279	12.7308	22.5263
64.3809	9.1557	8.0865	13.199	23.3127
66.6739	9.5037	8.3439	13.6757	24.1074
68.9931	9.8616	8.6002	14.1617	24.9112
71.3408	10.2301	8.8552	14.6576	25.7248
73.7194	10.6101	9.1088	15.1645	26.5492
76.1314	11.0026	9.3609	15.6831	27.3851
78.5796	11.4087	9.6113	16.2144	28.2336
81.0668	11.8296	9.8598	16.7595	29.0956
83.5959	12.2664	10.1062	17.3195	29.9721
86.1701	12.7208	10.3499	17.8958	30.8643
88.793	13.1942	10.5908	18.4896	31.7733
91.4683	13.6885	10.8283	19.1026	32.7005
94.1999	14.2053	11.0618	19.7362	33.6472
96.9928	14.7182	11.2896	20.363	34.6151
99.8973	11.8801	11.3909	17.5756	35.6218
Revenue (discounted)			762.5966	
Investment cost (discounted)			61.1145	
Total profit (discounted)			701.4821	

Table 4.22 – Impulse Control solutions for $\eta = 0.01$, where $T = 100$, $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
6.3504	1.9594	0	1.9594	3.2009
10.4003	2.3175	0.8717	2.7533	4.6045
13.8098	2.7062	1.3922	3.4023	5.7861
16.8941	3.0676	1.8360	3.9856	6.8550
19.7779	3.4110	2.2388	4.5304	7.8545
22.5261	3.7427	2.6147	5.0500	8.8070
25.1779	4.0670	2.9714	5.5527	9.7260
27.7594	4.3869	3.3135	6.0437	10.6207
30.2889	4.7047	3.6441	6.5267	11.4973
32.7803	5.0219	3.9655	7.0046	12.3608
35.2443	5.3400	4.2793	7.4796	13.2147
37.6894	5.6602	4.5867	7.9536	14.0622
40.1229	5.9836	4.8887	8.4279	14.9055
42.5509	6.3111	5.1860	8.9041	15.7470
44.9786	6.6436	5.4792	9.3832	16.5884
47.411	6.9821	5.7687	9.8664	17.4314
49.8523	7.3274	6.0550	10.3549	18.2775
52.3066	7.6804	6.3382	10.8495	19.1281
54.7778	8.0421	6.6186	11.3514	19.9845
57.2696	8.4135	6.8963	11.8616	20.8481
59.7857	8.7955	7.1713	12.3812	21.7201
62.3297	9.1893	7.4437	12.9112	22.6018
64.9055	9.5961	7.7133	13.4527	23.4945
67.5167	10.0172	7.9800	14.0072	24.3995
70.1673	10.4539	8.2437	14.5758	25.3181
72.8614	10.9080	8.5040	15.1600	26.2518
75.6034	11.3811	8.7606	15.7614	27.2021
78.3978	11.8752	9.0130	16.3817	28.1706
81.2498	12.3925	9.2606	17.0228	29.159
84.1647	12.9354	9.5028	17.6868	30.1692
87.1483	13.5069	9.7386	18.3762	31.2033
90.2071	14.1101	9.9670	19.0936	32.2634
93.3484	14.7483	10.1868	19.8417	33.3521
96.5805	15.3926	10.3954	20.5903	34.4723
99.9593	12.1273	10.4758	17.3652	35.6432
Revenue (discounted)			733.2291	
Investment cost (discounted)			56.6083	
Total profit (discounted)			676.6208	

Table 4.23 – Impulse Control solutions for $\eta = 0.02$, $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

τ_i	$I(\tau_i)$	$K(\tau_i^-)$	$K(\tau_i^+)$	$\theta(\tau_i^+)$
7.5126	2.2042	0	2.2042	3.6037
11.9020	2.6060	0.9162	3.0641	5.1249
15.5932	3.0359	1.4645	3.7681	6.4042
18.9345	3.4366	1.9315	4.4024	7.5622
22.0629	3.8188	2.3548	4.9962	8.6464
25.0493	4.1897	2.7494	5.5644	9.6814
27.9368	4.5542	3.1233	6.1158	10.6821
30.7539	4.9156	3.4815	6.6563	11.6585
33.5212	5.2765	3.8271	7.1901	12.6176
36.2541	5.6390	4.1625	7.7203	13.5647
38.9647	6.0049	4.4894	8.2496	14.5042
41.6632	6.3757	4.8090	8.7802	15.4394
44.3578	6.7529	5.1221	9.3139	16.3733
47.0561	7.1378	5.4295	9.8526	17.3084
49.7647	7.5319	5.7317	10.3977	18.2471
52.4898	7.9365	6.0290	10.9510	19.1916
55.2371	8.3532	6.3215	11.5139	20.1437
58.0124	8.7834	6.6095	12.0881	21.1056
60.8212	9.2288	6.8927	12.6752	22.0790
63.6691	9.6913	7.1711	13.2769	23.0660
66.5620	10.1729	7.4443	13.8951	24.0686
69.5058	10.6757	7.7120	14.5317	25.0889
72.5068	11.2023	7.9736	15.1891	26.1289
75.5718	11.7556	8.2282	15.8697	27.1912
78.7082	12.3386	8.4750	16.5761	28.2782
81.9241	12.9551	8.7129	17.3115	29.3927
85.2281	13.6094	8.9402	18.0795	30.5378
88.6304	14.3064	9.1552	18.8840	31.7170
92.1421	15.0518	9.3557	19.7296	32.9340
95.7764	15.8177	9.5378	20.5867	34.1936
99.5950	12.3688	9.5919	17.1648	35.5170
Revenue (discounted)			701.2148	
Investment cost (discounted)			52.6074	
Total profit (discounted)			648.6074	

Table 4.24 – Impulse Control solutions for $\eta = 0.03$, $T = 100$ and parameter values $r = 0.04$, $\delta = 0.2$, $b = \frac{1}{2}\log 2$, $\gamma = 0.5$, $\beta = 0.2$, $\alpha = 0$, $C = 2$, $K_0 = 0$ and $\theta(0) = 1$.

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CHAPTER 5

Numerical Algorithms for Deterministic Impulse Control models with applications

Abstract In this chapter we describe three different algorithms, two of which are new in the literature. We take both the size of the jump and the jump times as decision variables. The first (new) algorithm considers an Impulse Control problem as a (multipoint) Boundary Value Problem and uses a continuation technique to solve it. The second (new) approach is the continuation algorithm that requires the canonical system to be solved explicitly. This reduces the infinite dimensional problem to a finite dimensional system of, in general, nonlinear equations, without discretizing the problem. Finally, we present a gradient algorithm, where we reformulate the problem as a finite dimensional problem, which can be solved using some standard optimization techniques. As an application we solve a forest management problem and a dike heightening problem. We numerically compare the efficiency of our methods to other approaches, such as dynamic programming, backward algorithm and value function approach.

5.1 Introduction

For many problems in the area of economics and operations research it is realistic to allow for jumps in the state variable. Take, for example, a firm that increases the capital stock by a lumpy investment, or the decrease of the volume of a natural resource after each drilling. This chapter therefore considers optimal control models in which the time moment of these jumps and the size of the jumps are taken as (new) decision variables. Blaquière (1977a; 1977b; 1979; 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so called Impulse Control Maximum Principle, that gives necessary (and sufficient) optimality conditions for solving such problems. In Chapter 2 we present the necessary optimality conditions of the Impulse Control Maximum Principle based on the current value formulation. In Chapter 2 we also design a transformation, which ensures that the application of the Impulse Control Maximum Principle can be applied to problems with a fixed cost. For a review of the literature applying the impulse control maximum principle, we refer to Chapter 2 of this thesis.

Like Blaquière (1977a; 1977b; 1979; 1985) and Chahim et al. (2012b), we consider a framework where the number of jumps is not known. This distinguishes our approach from, e.g., Liu et al. (1998) and Wu and Teo (2006) where a gradient method is used assuming the number of jumps is known, and Augustin (2002, pp. 71–81) where the Impulse Control Maximum Principle is used for a fixed number of jumps (see e.g. Rempala (1990)). Other approaches in the literature include the value function approach found in Neuman and Costanza (1990), where a value function is defined for a fixed number of jumps and Erdlenbruch et al. (2011) or Eijgenraam et al. (2011) where dynamic programming is the tool of choice.

In the literature two different algorithms based on the Impulse Control Maximum Principle (Blaquière (1977a; 1977b; 1979; 1985) and Chahim et al. (2012b)) are derived. Luhmer (1986) derived a forward algorithm (starts at time 0) and Kort (1989, pp. 62–70) derived a backward algorithm (starts at final time horizon T). Both algorithms have some drawbacks. To initialize the forward algorithm the initial costate(s) value(s) is the choice variable. A similar drawback holds for the backward algorithm. Here information on the state variable(s) at the end of the planning period is needed, i.e. this (these) value(s) is (are) the choice variables(s).

In this chapter we describe three different algorithms, from which two (as far as we know) are new in the literature. We take both the size of the jump and the jump times as decision variables. The first (new) algorithm considers an Impulse Control problem as a (multipoint) Boundary Value Problem and uses a continuation technique to solve it. The second (new) approach is the continuation algorithm that requires the canonical system to be solved explicitly. This reduces the infinite dimensional problem to a finite dimensional system of, in general, nonlinear equations, without discretizing the problem. Finally, we present a gradient algorithm, where we reformulate the problem as a finite dimensional problem, which can be solved using some standard optimization techniques. As an application we solve a forest management problem and a dike heightening problem. We numerically compare the efficiency of our methods to other approaches, such as dynamic programming, backward algorithm and value function approach.

This chapter is organized as follows. In Section 5.2.1 we introduce the type of optimal control problem we consider in this chapter. In Section 5.3 we describe the three algorithm suitable for solving Impulse Control problems. In Section 5.3.1 we introduce some notation and show that the necessary conditions can be restated as a (multipoint) boundary value problem (BVP). Second, we describe the continuation algorithm in Section 5.3.2. Third, we describe the gradient algorithm in Section 5.3.3, which is developed by Hou and Wong (2011). In Section 5.4 we introduce two applications, one deals with forest management (Section 5.4.1), and one deals with dike heightening (Section 5.4.2). The numerical results for both applications are presented in Section 5.5. We compare our found results with the results found in the literature. Finally, in Section 5.6 we conclude and give recommendations for future research.

5.2 An Impulse Control Model

In this section we introduce a general Impulse Control model and provide necessary optimality conditions.

5.2.1 The Model

Let us denote x as the state variable, u as an ordinary control variable and v^i as the impulse control variable, where x and u are piecewise continuous functions of time[1]. We denote r as the discount rate leading to the discount factor e^{-rt} at time t. The terminal time or horizon date of the system or process is denoted by $T > 0$, and $x(T^+)$ stands for the state value immediately after a possible jump at time T. The profit of the system between jumps is given by $F(x, u, t)$, whereas $G(x, v, t)$ is the profit function associated with a jump, and $S(x(T^+))$ is the salvage value, i.e. the total costs or profit associated with the system after time T. Finally, $f(x, u, t)$ describes the continuous change of the state variable over time between the jump points and $g(x, v, t)$ is a function that represents the instantaneous (finite) change of the state variable when there is an impulse or jump.

[1]Note that the necessary optimality conditions presented in Theorem 5.2.1 also hold for measurable controls. Applications typically have piecewise continuous functions.

The above results in the following optimal control problem

$$\max_{u(\cdot),N,\tau_i,v^i} \left\{ \int_0^T \mathrm{e}^{-rt}\, F(x(t),u(t),t)\,\mathrm{d}t \right\} + \sum_{i=1}^{N} \mathrm{e}^{-r\tau_i}\, G(x(\tau_i^-),v^i,\tau_i) + \mathrm{e}^{-rT}\, S(x(T^+)), \tag{5.1a}$$

$$\text{s.t.}\quad \dot{x}(t) = f(x(t),u(t),t), \qquad \text{for} \quad t \in [0,T] \setminus \{\tau_1,\ldots,\tau_N\}, \tag{5.1b}$$

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-),v^i,\tau_i), \quad \text{for} \quad i \in \{1,\ldots,N\}, \tag{5.1c}$$

$$x(0^-) = x_0, \quad u(t) \in \mathcal{U}, \quad v^i \in \mathcal{V}, \quad i \in \{1,\ldots,N\}. \tag{5.1d}$$

For $N \in \mathbb{N}$ we assume the jump times to be sorted as

$$\tau_i \in [0,T] \quad \text{with} \quad 0 \le \tau_1 < \ldots < \tau_N \le T, \tag{5.1e}$$

$$x(\tau_i^+) = \lim_{t\downarrow\tau_i} x(t) \quad \text{and} \quad x(\tau_i^-) = \lim_{t\uparrow\tau_i} x(t), \quad \text{for} \quad i = 1,\ldots,N,$$

and

$$x_0 \in \mathbb{R}^n.$$

We assume that the domains $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{V} \subset \mathbb{R}^l$ are bounded convex sets. Further we impose that F, f, g and G are continuously differentiable in x on $\mathbb{R}^n$ and v^i on $\mathcal{V}$, $S(x)$ is continuously differentiable in x on $\mathbb{R}^n$, and that g and G are continuous in τ. Finally, when there is no jump, i.e. $v = 0$, we assume that

$$g(x,0,t) = 0,$$

for all x and t.

5.2.2 Necessary Optimality Conditions

We apply the Impulse Control Maximum Principle in current value formulation derived in Chahim et al. (2012b) to (5.1).[2] The resulting necessary optimality conditions are presented in Theorem 5.2.1.

Before we state Theorem 5.2.1, let us define the Hamiltonian $\mathcal{H}$ and the Impulse Hamiltonian $\mathcal{IH}$ as

$$\mathcal{H}(x,u,\lambda,t) := F(x,u,t) + \lambda f(x,u,t), \tag{5.2a}$$

$$\mathcal{IH}(x,v,\lambda,t) := G(x,v,t) + \lambda g(x,v,t), \tag{5.2b}$$

[2] Other references deriving the necessary optimality conditions for the Impulse Control problems are Blaquière (1977a; 1977b; 1979; 1985), Seierstad (1981) and Seierstad and Sydsæter (1987).

and define the following abbreviations

$$\mathcal{H}[s] := \mathcal{H}(x(s), u(s), \lambda(s), s), \tag{5.2c}$$

$$\mathcal{IH}[s, v] := \mathcal{IH}(x(s^-), v, \lambda(s^+), s), \tag{5.2d}$$

$$G[s, v] := G(x(s^-), v, s), \tag{5.2e}$$

$$g[s, v] := g(x(s^-), v, s). \tag{5.2f}$$

Theorem 5.2.1 (Impulse control maximum principle)**.**

Let for $N \in \mathbb{N}$ with $N > 0$ $(x^*(\cdot), u^*(\cdot), N, \tau_1^*, \dots, \tau_N^*, v^{1*}, \dots, v^{N*})$ be an optimal solution of (5.1). Then there exists a (piecewise absolutely continuous) adjoint variable $\lambda(\cdot)$ such that the following conditions hold:

$$u^*(t) \in \underset{u}{\operatorname{argmax}}\, \mathcal{H}(x^*(t), u(t), \lambda(t), t), \quad t \in [0, T], \tag{5.3a}$$

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial}{\partial x}\mathcal{H}(x^*(t), u^*(t), \lambda(t), t), \quad t \in [0, T] \setminus \{\tau_1^*, \dots, \tau_N^*\}. \tag{5.3b}$$

For every $t = \tau_i^*,\ (i = 1, \dots N)$, we have

$$\frac{\partial}{\partial v}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*)(v - v^{i*}) \le 0, \quad v \in \mathcal{V}, \tag{5.3c}$$

$$\lambda(\tau_i^{*+}) - \lambda(\tau_i^{*-}) = -\frac{\partial}{\partial x}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*), \tag{5.3d}$$

$$\mathcal{H}[\tau_i^{*+}] - \mathcal{H}[\tau_i^{*-}] + rG[\tau_i^*, v^{i*}] - \frac{\partial}{\partial \tau}\mathcal{IH}[\tau_i^*, v^{i*}] \begin{cases} > 0 & \tau_i^* = 0 \\ = 0 & \tau_i^* \in (0, T) \\ < 0 & \tau_i^* = T. \end{cases} \tag{5.3e}$$

For $t \in [0, T] \setminus \{\tau_1^*, \dots, \tau_N^*\}$ it holds that

$$\frac{\partial}{\partial v}\mathcal{IH}(x^*(t), 0, \lambda(t), t)v \le 0, \quad v \in \mathcal{V}. \tag{5.3f}$$

The transversality condition is

$$\lambda(T^+) = \frac{\partial}{\partial x}S(x^*(T^+)). \tag{5.3g}$$

Proof: See Blaquière (1977a; 1985). ■

To simplify the presentation and to concentrate on the main concepts of the numerical algorithm, besides the earlier assumptions, we further make the following assumptions.

Assumption 5.2.1. For every time horizon $T \ge 0$ there exists a unique optimal solution of (5.1), with a finite number of jumps (which in general depends on T).

This assumption is needed for the boundary value problem approach and the continuation algorithm. If this assumption does not hold, both algorithms will not generate a solution since the number of jumps is not finite. This assumption is not required for the gradient algorithm, since the number of jumps is fixed.

Assumption 5.2.2. Let for $T > 0$ the jump times be $(\tau_i)_{i=1}^N$ with $0 < \tau_1 < \ldots < \tau_N < T$, and $\bar{x}(T) := (x(\tau_1^-), x(\tau_1^+), v_1, \ldots, x(\tau_N^-), x(\tau_N^+), v_N)$ be the vector of left and right limits of the states together with the optimal impulse control values for the given time horizon T. Then in a neighborhood of T the solution vector $\bar{x}(T)$ is continuous.

We need this assumption again for both the boundary value problem approach and the continuation algorithm. For both algorithms T is a continuation variable. During the continuation process T is increased and the conditions for possible jumps are monitored.

Assumption 5.2.3. The model does not include a continuous control.

For simplicity we state this assumption. Then the boundary value problem approach is still a suitable method to solve the problem. The gradient method and the continuation algorithm depend on whether the system is explicitly solvable or not.

Assumption 5.2.4. Condition (5.3c) together with Assumption 5.2.3 implies

$$\frac{\partial}{\partial v}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*) = 0, \tag{5.4}$$

and with $\frac{\partial^2}{\partial v^2}\mathcal{IH}(x^*(\tau_i^{*-}), v^{i*}, \lambda(\tau_i^{*+}), \tau_i^*) < 0$ this yields

$$v^{i*} = v(x^*(\tau_i^{*-}), \lambda(\tau_i^{*+}), \tau_i^*). \tag{5.5}$$

In general condition (5.3c) does not imply that the optimal impulse control value can be found as the arg max of the Impulse Hamiltonian. For simplicity we restrict ourself to such function in this chapter.

5.3 Numerical Algorithms

In this section we describe three different algorithms to solve Impulse Control problems. We state a (multipoint) boundary value problem for Impulse Control problems in Section 5.3.1 which is (as far as we know) new in the literature, describe the gradient method approach developed by Hou and Wong (2011) in Section 5.3.3, and finally we describe a second new approach that we call the continuation algorithm in Section 5.3.2.

5.3.1 (Multipoint) Boundary Value Approach

In this section we describe a (multipoint) boundary value problem (BVP), that is useful to solve Impulse Control problems. The idea behind the boundary value approach is that between two jumps the system of differential equations (canonical system) combined with the boundary conditions (initial and final conditions) is solved. After each found jump the (multipoint) BVP is updated to find the next jump.

To formulate the (multipoint) BVP we introduce the following notation for the canonical system dynamics:

$$\dot{x}(t) = h_1(x(t), \lambda(t), t), \tag{5.6a}$$

$$\dot{\lambda}(t) = h_2(x(t), \lambda(t), t). \tag{5.6b}$$

For the conditions at a jumping time τ we define:

$$j^x(x(\tau^+), x(\tau^-), \lambda(\tau^+), \tau) := x(\tau^+) - x(\tau^-) - g[\tau, x(\tau^+) - x(\tau^-)], \tag{5.6c}$$

$$j^\lambda(x(\tau^-), \lambda(\tau^+), \lambda(\tau^-), \tau) := \lambda(\tau^+) - \lambda(\tau^-) + \frac{\partial}{\partial x}\mathcal{IH}[\tau, x(\tau^+) - x(\tau^-)], \tag{5.6d}$$

$$\begin{aligned} j^\tau(x(\tau^-), x(\tau^+), \lambda(\tau^+), \lambda(\tau^-), \tau) := \mathcal{H}[\tau^+] - \mathcal{H}[\tau^-] + \\ rG[\tau, v] - \frac{\partial}{\partial \tau}\mathcal{IH}[\tau, x(\tau^+) - x(\tau^-)]. \end{aligned} \tag{5.6e}$$

Now let $(x^*(\cdot), u^*(\cdot), N, \tau_1^*, \ldots, \tau_N^*, v^{1*}, \ldots, v^{N*})$ be the optimal solution of (5.1) with $0 < \tau_1^* < \ldots < \tau_N^* < T$. Then the necessary conditions yield the following (multipoint) BVP:

$$\dot{x}_i(t) = h_1(x_i(t), \lambda_i(t), t), \quad t \in [\tau_{i-1}, \tau_i],\ i = 1, \ldots, N+1, \tag{5.7a}$$

$$\dot{\lambda}_i(t) = h_2(x_i(t), \lambda_i(t), t), \quad t \in [\tau_{i-1}, \tau_i],\ i = 1, \ldots, N+1, \tag{5.7b}$$

$$j^x(x_i(\tau_i^+), x_i(\tau_i^-), \lambda_i(\tau_i^+), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{5.7c}$$

$$j^\lambda(x_i(\tau_i^-), \lambda_i(\tau_i^+), \lambda_i(\tau_i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{5.7d}$$

$$j^{\tau_i}(x_i(\tau_i^-), x_i(\tau_i^+), \lambda_i(\tau_i^+), \lambda_i(\tau_i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{5.7e}$$

$$\mathcal{S}(x_{N+1}(T), \lambda_{N+1}(T)) = 0, \tag{5.7f}$$

$$x_1(0) - x_0 = 0, \tag{5.7g}$$

where (5.7f) denotes the transversality condition (5.3g), $\tau_0 = 0$ and $\tau_{N+1} = T$.

After defining $t(s) := \tau_i - (i - s)\Delta\tau_i$, with $\Delta\tau_i := \tau_i - \tau_{i-1}$, we rewrite (5.7) into

$$\dot{x}_i(s) = \Delta\tau_i h_1(x_i(s), \lambda_i(s), t(s)), \quad s \in [i-1, i],\ i = 1, \ldots, N+1, \tag{5.8a}$$

$$\dot{\lambda}_i(s) = \Delta\tau_i h_2(x_i(s), \lambda_i(s), t(s)), \quad s \in [i-1, i],\ i = 1, \ldots, N+1, \tag{5.8b}$$

$$j^x(x_i(i^+), x_i(i^-), \lambda_i(i^+), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{5.8c}$$

$$j^\lambda(x_i(i^-), \lambda_i(i^+), \lambda_i(i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{5.8d}$$

$$j^i(x_i(i^-), x_i(i^+), \lambda_i(i^+), \lambda_i(i^-), \tau_i) = 0, \quad i = 1, \ldots, N, \tag{5.8e}$$

$$\mathcal{S}(x_{N+1}(N+1), \lambda_{N+1}(N+1)) = 0, \tag{5.8f}$$

$$x_1(0) - x_0 = 0. \tag{5.8g}$$

The jump times $\tau_i,\ i = 1\ldots, N$, appear as unknown variables.

To handle the case $\tau_N = T$ we introduce the (unknown) variables

$$\mathrm{xT} := x_{N+1}(T^+),$$
$$\mathrm{lT} := \lambda_{N+1}(T^+),$$

together with the additional boundary conditions

$$j^x(\mathrm{xT}, x_{N+1}(N+1), \mathrm{lT}, T) = 0, \tag{5.9a}$$
$$j^\lambda(x_{N+1}(N+1), \mathrm{lT}, \lambda_{N+1}(N+1), T) = 0, \tag{5.9b}$$

and replace (5.8f) by

$$\mathcal{S}(\mathrm{xT}, \mathrm{lT}) = 0. \tag{5.9c}$$

The case $\tau_1 = 0$ can be treated in an analogous way. We therefore set

$$\mathrm{x0} := x_1(0^+),$$
$$\mathrm{l0} := \lambda_1(0^+),$$

together with the additional boundary conditions

$$j^x(\mathrm{x0}, x_0, \mathrm{l0}, 0) = 0, \tag{5.10a}$$
$$j^\lambda(x_0, \mathrm{l0}, \lambda_1(0), 0) = 0, \tag{5.10b}$$

and replace (5.8g) by

$$x_1(0) - \mathrm{x0} = 0. \tag{5.10c}$$

During the continuation process it may be of interest to determine the exact value of end time T where the solution jumps at the end time and additionally the condition (5.8e) is satisfied. In general this characterizes the crossing from a jump at the boundary to an interior jump. For that case the time horizon T is considered as a free variable and the condition

$$j^{N+1}(x_{N+1}(N+1), \mathrm{xT}, \lambda_{N+1}(N+1), \mathrm{lT}, T) = 0, \tag{5.11}$$

is appended to (5.9).

Initializing the BVP

To find the solution of a specific problem of type (5.1) we can apply a continuation strategy with respect to the time horizon T. Therefore, as a first step we have to determine an initial (optimal) solution.

Due to Assumption 5.2.1, the initial condition together with the transversality condition yield the necessary equations for $T=0$. This solution can be used as a starting point for paths, which for a "small" time horizon do not exhibit a jumping point.

5.3.2 Continuation Algorithm

Let us consider the initial value problem (IVP) (5.8a) and (5.8b) on the time interval $[i-1,i]$ with

$$\dot{y}(s) = \Delta\tau_i h_1(y(s),\mu(s),t(s)), \quad s \in [i-1,i], \tag{5.12a}$$

$$\dot{\mu}(s) = \Delta\tau_i h_2(y(s),\mu(s),t(s)), \quad s \in [i-1,i]. \tag{5.12b}$$

With initial conditions

$$y(i-1) = x(\tau_i),\ \mu(i-1) = \lambda(\tau_i), \tag{5.12c}$$

the solution can formally be written as

$$y(i) - y(i-1) = \Delta\tau_i \int_{i-1}^{i} h_1(y(s),\mu(s),t(s))\,\mathrm{d}s,$$

$$\mu(i) - \mu(i-1) = \Delta\tau_i \int_{i-1}^{i} h_2(y(s),\mu(s),t(s))\,\mathrm{d}s,$$

or even more general as an implicit equation

$$F(y(i-1),\mu(i-1),y(i),\mu(i),\tau_{i-1},\tau_i) = 0.$$

To simplify notation, we introduce the following notation:

$$y_{2i} := \begin{pmatrix} x(\tau_i^-) \\ \lambda(\tau_i^-) \end{pmatrix} \quad y_{2i+1} := \begin{pmatrix} x(\tau_i^+) \\ \lambda(\tau_i^+) \end{pmatrix}, \quad i = 0,1,\ldots,N.$$

Then the system (5.8) can be stated as

$$\Omega_0(y_0,y_1,\tau_0) = 0 \in \mathbb{R}^{3n}, \tag{5.13a}$$

$$\Omega_{N+1}(y_{2N},y_{2N+1},\tau_{N+1}) = 0 \in \mathbb{R}^{3n}, \tag{5.13b}$$

$$\Omega_i = \Upsilon(y_{2i},y_{2i+1},\tau_i) = 0 \in \mathbb{R}^{2n+1}, \quad i = 1,\ldots,N, \tag{5.13c}$$

$$\Gamma_i = F(y_{2i+1},y_{2(i+1)},\tau_i,\tau_{i+1}) = 0 \in \mathbb{R}^{2n},\ i = 0,1,\ldots,N, \tag{5.13d}$$

where (5.13a) denotes the initial condition, (5.13b) the transversality condition, (5.13c) the connecting condition for interior jumping points, and (5.13d) the solution of the IVP. Thus in total we have $8n + N(4n+1)$ equations ((5.13a) generates $3n$ equations, (5.13b) also generates $3n$ equations, (5.13c) generates $N(2n+1)$ equations, and finally (5.13d) generates $(N+1)2n$ equations) and the same number of unknowns $(y_0, \ldots, y_{2(N+1)+1}, \tau_1, \ldots, \tau_N)$ $(y_0, \ldots, y_{2(N+1)+1}$ are $2n(2(N+1)+2)$ variables and $\tau_1, \ldots, \tau_N$ are N variables, gives a total of $8n + N(4n+1)$ variables). Then

$$\Omega = [\Omega_0 \quad \Omega_1 \ldots \Omega_N]' \in \mathbb{R}^{8n+N(2n+1)}, \tag{5.14a}$$

$$\Gamma = [\Gamma_0 \quad \Gamma_1 \ldots \Gamma_{N+1}]' \in \mathbb{R}^{2nN}. \tag{5.14b}$$

If the IVP (5.12) can be solved explicitly, the formulation (5.14) has the advantage of reducing the infinite dimensional problem to a finite dimensional system of, in general, nonlinear equations, without discretizing the problem.

5.3.3 Gradient Algorithm

If the dynamics (5.1b) and the integral part of the objective function (5.1a) are simple enough to solve them explicitly, then the problem can be restated (without numerical discretization) as a finite dimensional problem. This can then be solved by some standard optimization algorithm, e.g. the numerical optimizer `fmincon` under MATLAB.

Problem (5.1) can be written as

$$\begin{aligned} &\max_{N,\tau_i,v^i} \sum_{i=0}^{N} \Gamma(x(\tau_i^+), x(\tau_{i+1}^-), t_i, t_{i+1}) + \\ &\quad \sum_{i=1}^{N} \mathrm{e}^{-r\tau_i} G(x(\tau_i^-), v^i, \tau_i) + \mathrm{e}^{-rT} S(x(T^+)), \quad i = 0, \ldots, N, \end{aligned} \tag{5.15a}$$

$$\text{s.t.} \quad x(t_{i+1}^-) = \Phi(x(t_i^+), t_i, t_{i+1}), \qquad \text{for} \quad i = 0, \ldots, N, \tag{5.15b}$$

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v^i, \tau_i), \quad \text{for} \quad i = 1, \ldots, N, \tag{5.15c}$$

$$x(0^-) = x_0 \in \mathbb{R}^n, \tag{5.15d}$$

with

$$\tau_k = t_k, \quad k \in \{0, 1, \ldots, N, N+1\}, \quad t_{N+1} = T, \tag{5.15e}$$

$$\Gamma(x(t_i^+), x(t_{i+1}^-), t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} \mathrm{e}^{-rt} F(x(t), t)\, \mathrm{d}t, \tag{5.15f}$$

$$\Phi(x(t_i^+), t_i, t_{i+1}) = x(t_i^+) + \int_{t_i}^{t_{i+1}} f(x(t), t)\, \mathrm{d}t. \tag{5.15g}$$

Setting

$$y = (x(t_0^-), x(t_0^+), \ldots, x(T^-), x(T^+), v^1, \ldots, v^N, \tau_1, \ldots, \tau_N)', \tag{5.16}$$

problem (5.15) becomes a finite dimensional maximization problem. To keep the notation simple, in a first step we subsequently assume that the jumps only occur within the interior of the interval $[t_0,\ T]$. Therefore $\tau_k = t_k,\ k = 1, \ldots, N$, and $y \in \mathbb{R}^{4N+4}$ (i.e. y consists of $N+2$ left and $N+2$ right limits, N jumps, and N jump times). In that case the doubling (left and right limit) of the initial and end state is superfluous but allows an immediate generalization in case that a jump also occurs at the beginning or the end.

Next we derive the necessary optimality conditions, which, of course, reproduce the necessary optimality conditions from the Impulse Control Maximum Principle. First we start with the derivatives (gradients) of the equality constraints (5.15b)-(5.15d). In the new coordinates y_i these constraints become

$$
\begin{aligned}
&c_1 = y_1 - x_0 = 0,\\
&c_{2+k} = y_{2k+1} - y_{2k} - \Phi(y_{2k}, y_{2(N+2)+N+k}, y_{2(N+2)+N+k+1}) = 0,\ k = 0, \ldots, N,\\
&c_{2+N+1+k} = y_{2(k+1)} - y_{2k+1} = 0,\ k = 0, N+1,\\
&c_{2+N+1+k} = y_{2(k+1)} - y_{2k+1} - g(y_{2k+1}, y_{2(N+2)+k}, y_{2(N+2)+N+k}) = 0,\ k = 1, \ldots, N.
\end{aligned}
$$

Therefore the derivatives are calculated as

$$
\begin{aligned}
&\frac{\partial c_1}{\partial y_1} = 1,\\
&\frac{\partial c_{2+k}}{\partial y_{2k+1}} = 1,\\
&\frac{\partial c_{2+k}}{\partial y_{2k}} = -1 - \partial^{(1)}\Phi,\\
&\frac{\partial c_{2+k}}{\partial y_{2(N+2)+N+k}} = -\partial^{(2)}\Phi,\\
&\frac{\partial c_{2+k}}{\partial y_{2(N+2)+N+k+1}} = -\partial^{(3)}\Phi,\\
&\frac{\partial c_{2+N+1+k}}{\partial y_{2(k+1)}} = 1, \quad k = 0, \ldots, N+1,\\
&\frac{\partial c_{2+N+1+k}}{\partial y_{2k+1}} = -1, \quad k = 0,\ N+1,\\
&\frac{\partial c_{2+N+1+k}}{\partial y_{2k+1}} = -1 - \partial^{(1)}g, \quad k = 1, \ldots, N,\\
&\frac{\partial c_{2+N+1+k}}{\partial y_{2(N+2)+k}} = -\partial^{(2)}g, \quad k = 1, \ldots, N,\\
&\frac{\partial c_{2+N+1+k}}{\partial y_{2(N+2)+N+k}} = -\partial^{(3)}g, \quad k = 1, \ldots, N,
\end{aligned}
$$

where $\partial^{(i)}$ denotes the partial derivative of a function with respect to its i-th argument. Rewriting the objective function (5.15a) in the coordinates y we find

$$\begin{aligned} V(y) &= \Gamma(y_1, y_2, y_{2(N+2)+N+1}, y_{2(N+2)+N+2}) \\ &\quad + \sum_{i=1}^{N} \Gamma(x(t_i^+), x(t_{i+1}^-), t_i, t_{i+1}) \\ &\quad + \sum_{i=1}^{N} \mathrm{e}^{-r\tau_i}\, G(x(\tau_i^-), v^i, \tau_i) + \mathrm{e}^{-rT}\, S(x(T^+)), \end{aligned}$$

and the derivatives are given as

$$\frac{\partial V(y)}{\partial y_1} = 1.$$

For a thorough discussion and motivation we refer to Hou and Wong (2011).

In order to find the optimal solution using the gradient algorithm we need some information about the structure of the problem, i.e. have some knowledge about the optimal number of jumps. Neuman and Costanza (1990) use the value function approach and assume that for each initial state, the value function V is well behaved, in the sense that there is an index k such that V_k (where V_k denotes the value function having k jumps) is greater than other V_i, i.e. V_is are nondecreasing for $i \leq k$ and monotonically decreasing for $i \geq k$. The main reason for this assumption is that this guarantees that only a finite number of steps is necessary to achieve the optimum.

To overcome this problem we use the solution provided by the continuation algorithm to initialize the gradient method approach. From numerical experiments we know that the continuation algorithm has provided the same (optimal) solution for impulse control problems solved using the backward algorithm, dynamic programming, or the value function approach. We have no proof that the algorithm converges or finds the optimal solution for all Impulse Control problems.

5.4 Two Applications

5.4.1 A Forest Management Model

To exemplify the numerical techniques we use a model described in Neuman and Costanza (1990) where the optimal solution for forest management is derived using impulse control. It consists, at time t, of one state $w(t) \in \mathbb{R}_+$ denoting the size of the forest and one impulse control $z \in \mathbb{R}_+$ denoting the size of the cut (of the forest). The dynamics of the forest is described by a logistic term $g(y(t))$. Forest growth is then presented by

$$\dot{w}(t) = g(w(t)) := w(t)(a - bw(t)), \qquad t \geq 0,$$

with a and b positive constants. At time zero the size of the forest is equal to some initial value, i.e.

$$w(0) = x \geq 0.$$

When management is imposed on forest evolution, the forest is cut at times $\tau_i \in \mathbb{R}_+$ $(i = 1, \dots N)$ with N the number of cuts such that the size of the forest changes by:

$$w(\tau_i^+) - w(\tau_i^-) = z^i, \quad \text{for} \quad i \in \{1, \dots, N\}.$$

The total benefit generated by the dynamic system is given by

$$q(x) + \int_0^T f(w(s), s)e^{-rt}dt + \sum_{i=1}^{N} k(w(\tau_i), \tau_i, z^i)e^{-r\tau_i} + p(w(T^+)e^{-rT},$$

where $q(x)$ is the initial cost function, $f(w,t)$ is the profit function of the system per unit time, and $k(w, \tau_i, z^i)$ is the cost of the impulse z^i applied to the state $w(\tau_i)$ at time τ_i.

The impulse cost function is given by

$$k(w, \tau, z) = D + K(w, z) = D - g_0 z + g_1 z^2 \quad \text{for } z > 0,$$

where $D < 0$ can be considered as a fixed cost for cutting the forest and $K(w, z)$ being the variable profit generated by cutting the forest, g_0 and g_1 are some positive constants. If $z = 0$ we assume that $k(w, \tau, 0) = 0$. The initial cost function is given by

$$q(x) = -q_0(x - x_0),$$

where q_0 is a positive constant and x_0 is some bound imposed on the states, due to either ecological or practical constraints. The profit of the system is given by

$$f(w, t) = f_0,$$

with f_0 some positive constant. Finally, the salvage value is defined as

$$p(w(T^+)) = g_0(w(T^+) - x_0) - g_1(w(T^+) - x_0)^2.$$

Summing up, the optimal control problem can be written as

$$\max_{N,\tau,z}\Bigg\{-q_0(x-x_0)+\int_0^T \mathrm{e}^{-rt} f_0 dt$$
$$+\sum_{i=1}^{N}\mathrm{e}^{-r\tau_i}(D-g_0z^i)+\mathrm{e}^{-rT}(g_0(w(T^+)-x_0)-g_1(w(T^+)-x_0)^2)\Bigg\}, \tag{5.17a}$$

$$\text{s.t.}\quad \dot{w}(t)=w(t)(a-bw(t)),\quad \text{for}\quad t\in[0,T]\setminus\{\tau_1,\dots,\tau_N\}, \tag{5.17b}$$
$$w(\tau_i^+)-w(\tau_i^-)=z^i,\qquad \text{for}\quad i\in\{1,\dots,N\}, \tag{5.17c}$$
$$w(0^-)=x\ge 0, \tag{5.17d}$$
$$w(t)\in\mathbb{R}_+,\quad z^i\in(-\infty,0],\quad 0\le\tau_1<\tau_2<\ldots<\tau_N\le T, \tag{5.17e}$$

where r denotes the discount rate. For the analysis of this model the Impulse Control Maximum Principle is used, where the details are presented in Appendix 5A.1.

5.4.2 Dike Heightening Problem

This section describes a problem taken from Chahim et al. (2012a) where the optimal timing of the heightening of a dike is studied. The cost-benefit-economic decision problem contains two types of cost, namely investment cost and cost due to damage (caused by failure of protection by the dikes). It consists, at time t, of one state $H(t)\in\mathbb{R}_+$ denoting the height of the dike relative to the initial situation, i.e. $H(0)=0$ (cm) and one impulse control variable v^i denoting the i-th dike heightening of the dike. It is assumed that between two heightenings the dike height does not change, i.e. the dynamics of the dike are presented by

$$\dot{H}(t)=0.$$

The dike increases at times $\tau_i\in\mathbb{R}_+$ $(i=1,\dots N)$, with N the number of heightenings such that the height of the dike is increased by

$$H(\tau_i^+)-H(\tau_i^-)=v^i,\quad \text{for}\quad i\in\{1,\dots,N\}.$$

The objective consists of two parts. The first part is the total (discounted) expected damage cost, which is given by

$$\int_0^T S(t)e^{-rt}dt+\frac{S(T)e^{-rT}}{r},$$

where $S(t)$ denotes the expected damage at time t, i.e. $S(t)=P(t)V(t)$, where $P(t)$ stands for the flood probability and $V(t)$ the damage of a flood (million €) at time

t. The flood probability $P(t)$ (1/year) in year t is defined as

$$P(t) = P_0 e^{\alpha\eta t} e^{-\alpha H(t)}, \tag{5.18}$$

where α (1/cm) stands for the parameter in the exponential distribution regarding the flood probability, η (cm/year) is the parameter that indicates the increase of the water level per year, and P_0 denotes the flood probability at $t = 0$. The damage of a flood $V(t)$ (million €) is given by

$$V(t) = V_0 e^{\gamma t} e^{\zeta H(t)}, \tag{5.19}$$

in which γ (per year) is the parameter for economic growth, and ζ (1/cm) stands for the damage increase per cm dike height. V_0 (million €) denotes the loss by flooding at time $t = 0$. The second part of the objective is the total (discounted) investment cost

$$\sum_{i=1}^{N} I(v^i, H(\tau_i^-))e^{-r\tau_i},$$

where $H(\tau^-)$ denotes the height of the dike (in cm) just before the dike update at time τ (left-limit of $H(t)$ at $t = \tau$). The investment cost is given by

$$I(v^i, H(\tau^-)) = \begin{cases} a_0(H(\tau^-) + v^i)^2 + b_0 v^i + c_0 & \text{for } v^i > 0 \\ 0 & \text{for } v^i = 0, \end{cases}$$

for suitably chosen constants a_0, b_0 and c_0. Summing up, the Impulse Control model can be written as

$$\min_{v,\tau,N} \left\{ \int_0^T S(t)e^{-rt}dt + \sum_{i=1}^{N} I(v^i, H(\tau_i^-))e^{-r\tau_i} + e^{-rT}\frac{S(T)}{r} \right\}, \tag{5.20a}$$

$$\text{s.t.} \quad \dot{H}(t) = 0, \qquad \text{for} \quad t \in [0, T] \setminus \{\tau_1, \ldots, \tau_N\}, \tag{5.20b}$$

$$H(\tau_i^+) - H(\tau_i^-) = v^i, \quad \text{for} \quad i \in \{1, \ldots, N\}, \tag{5.20c}$$

$$H(0^-) = 0, \tag{5.20d}$$

$$H(t) \in \mathbb{R}_+, \quad v^i \in [0, \infty), \quad 0 \le \tau_1 < \tau_2, \ldots < \tau_N \le T. \tag{5.20e}$$

For the analysis of this model the impulse control maximum principle is used, where the details are carried out in Appendix 5A.2. For an extensive description of the model we refer to Chahim et al. (2012a).

5.5 Numerical Results

In this section we present results for two different applications using the continuation algorithm and make a comparison with results derived using other approaches.

5.5.1 The Forest Model

In this section we present the results for the optimal forest management problem described in the previous section. The parameter value presented in Table 5.1 are taken from Neuman and Costanza (1990).

r	a	b	D	f_0	g_0	g_1	q_0	x_0	y_0	T
0.05	0.2059	0.00344	-190	-15	24.5	0	40	5	34.4	8

Table 5.1 – Parameter values for the optimal forest management model.

τ	z_i	$w(\tau^-)$	$w(\tau^+)$
0	0	34.4	34.4
1	-24.2	37.35	13.08
8	0	32.43	32.43
Discounted revenue		-441.1751	

Table 5.2 – Result of value function approach found in Neuman and Costanza (1990).

τ	z_i	$w(\tau^-)$	$w(\tau^+)$
0	0	34.400	34.400
0.8216	-23.5757	36.8383	13.2626
8	0	33.290	33.290
Discounted revenue		-438.2973	

Table 5.3 – Result of the continuation algorithm.

The results we derive using the continuation algorithm are presented in Table 5.3. The results of Table 5.3 are similar to the results found in Neuman and Costanza (1990) presented in Table 5.2. The continuation algorithm (same holds for BVP algorithm) has two advantages over the value function approach described in Neuman and Costanza (1990). First, we do not have to discretize the time horizon. This results in a better objective value and hence a better solution to the original problem. In Figure 5.1 we plot the size of the forest as a function of time. Initially, the size of the forest increases, then at a some time instance the forest is cut. Hence, the size of the forest jumps downward and then grows again. Second, we did not have to solve the problems for different number of cuts to find the optimal solution to our forest management problem.

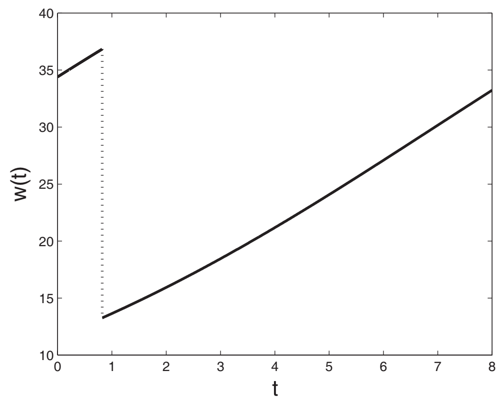

Figure 5.1 – Dynamics of the forest over time.

5.5.2 The Dike Heightening Model

In this section we present the optimal solution for a dike. The parameter values presented in Table 5.4 are taken from Eijgenraam et al. (2011).

In Table 5.5 the solution for three different approaches are presented. In the

a_0	b_0	c_0	V_0	r	P_0
0.0014	0.6258	16.6939	1564.9	0.04	1/2270

H_0	α	η	γ	ζ	T
0	0.33027	0.32	0.02	0.003774	300

Table 5.4 – Parameter values for dike 10.

second column the results for the continuation algorithm are given, the third column presents the results found by the backward algorithm used in Section 3.3, and in the fourth column the results for dynamic programming (DP) are given taken from Eijgenraam et al. (2011).

Unlike dynamic programming, both the continuation algorithm and the backward algorithm do not need to discretize time. However, for the initialization of the backward algorithm, we need the discretization of the state at the end of the time horizon (final stage), $H(T)$, and dynamic programming requires the discretization of time and of the heights (states) for each stage. The continuation algorithm does not need any input on the state variable $H(T)$. Even though the solutions for the backward algorithm and the continuation algorithm are similar, the continuation algorithm (same holds for the BVP approach) finds the optimal solution without running the algorithm for different end heights $H(T)$. In Chahim et al. (2012a) the authors discretize the state variable as is required for the dynamic programming approach in Eijgenraam et al. (2011) and take that $H(T)$ that minimizes (5.20a).

Approach[a]	BA	DP	CA
$(\tau_i : u_i)$	272.8 : 52.18	274 : 51.84	272.7 :52.21
	217.0 : 56.43	219 : 55.68	217.0 :56.45
	160.1 : 56.90	162 : 57.60	160.0:56.90
	103.0 : 56.95	104 : 57.60	103.0 :56.96
	45.9 : 56.96	46 : 57.60	45.8 :56.96
$H(T)$	279.41	280.32	279.48
Total cost	40.03	40.04	40.03

[a] Backward algorithm (BA), dynamic programming (DP), and continuation algorithm (CA)

Table 5.5 – Results for dike 10.

5.6 Conclusions and Recommendations

We describe three different numerical methods to solve Impulse Control problems. The first (new) algorithm considers an Impulse Control problem as a (multipoint) Boundary Value Problem and uses a continuation technique to solve it. The second (new) approach is the continuation algorithm that requires the canonical system to be solved explicitly. This reduces the infinite dimensional problem to a finite dimensional system of, in general, nonlinear equations, without discretizing the problem. The third algorithm is a gradient algorithm, where the problem is reformulated as a finite dimensional problem, which can be solved using some standard optimization techniques. We use the continuation algorithm to solve the optimal forest management problem (same results found for the boundary value problem approach) and the dike heightening problem. Although numerical results found by the continuation algorithm (same holds for the boundary value problem approach) are at least as good as the results found in the literature, a formal proof that the boundary value problem approach and the continuation algorithm provide the optimal solution is subject for future research.

Appendix 5A Necessary Optimality Conditions for the Applications

5A.1 The Forest Management Model

Let us define the current value Hamiltonian

$$\mathcal{H}(w, \lambda, t) := f_0 + \lambda w(a - bw), \tag{5.21}$$

and the current value Impulse Hamiltonian

$$\mathcal{IH}(z, \lambda, t) := D - g_0 z + g_1 z^2 + \lambda z. \tag{5.22}$$

We obtain the adjoint equation

$$\dot{\lambda}(t) = (r - a + 2bw(t))\lambda(t), \quad \text{for} \quad t \neq \tau_i, i = 1, \ldots, N, \tag{5.23}$$

with the transversality condition

$$\lambda(T) = g_0 - 2g_1 w(T^-). \tag{5.24}$$

The jump conditions are

$$-g_0 + g_1 z^i + \lambda(\tau_i^+) = 0, \quad \text{for } i = 1, \ldots, N, \tag{5.25}$$

and

$$\lambda(\tau_i^+) - \lambda(\tau_i^-) = 0, \quad \text{for } i = 1, \ldots, N, \tag{5.26}$$

from which we can conclude that the costate $\lambda(t)$ is continuous at every jump point. The condition for determining the optimal switching time τ_i is

$$\begin{aligned} &\lambda(\tau_i^+)w(\tau_i^+)(a - bw(\tau_i^+)) - \lambda(\tau_i^-)w(\tau_i^-)(a - bw(\tau_i^-)) \\ &\qquad + rD - rg_0 z^i + rg_1 {z^i}^2 \begin{cases} > 0 & \text{if } \tau_i = 0 \\ = 0 & \text{if } \tau_i \in (0, T) \\ < 0 & \text{if } \tau_i = T. \end{cases} \end{aligned} \tag{5.27}$$

5A.2 The Dike Heightening Model

Let us define the current value Hamiltonian

$$\mathcal{H}(t, H) = -S_0 e^{\beta t} e^{-\theta H}, \tag{5.28}$$

and the current value Impulse Hamiltonian

$$\mathcal{IH}(H, v, \lambda, t) = -I(v, H) + \lambda v = -A_0(H + v)^2 \\ -b_0 v - c_0 + \lambda v, \tag{5.29}$$

and obtain the adjoint equation

$$\dot{\lambda}(t) = r\lambda(t) - \theta S_0 e^{\beta t} e^{-\theta H(t)}, \quad \text{for} \quad t \neq \tau_i, i = 1, \ldots, N, \tag{5.30}$$

with the transversality condition

$$\lambda(T) = \frac{\theta S_0 e^{\beta T} e^{-\theta H(T)}}{r}. \tag{5.31}$$

The jump conditions are

$$-I_u\left(u_i, H\left(\tau_i^-\right)\right) + \lambda\left(\tau_i^+\right) = 0, \quad \text{for } i = 1, \ldots, N, \tag{5.32}$$

$$\lambda(\tau_i^+) - \lambda(\tau_i^-) == I_H\left(u_i, H(\tau_i^-)\right), \quad \text{for } i = 1, \ldots, N \tag{5.33}$$

The condition for determining the optimal switching time τ_i is

$$S_0 e^{\beta\tau_i}\left(e^{-\theta H(\tau_i^-)} - e^{-\theta H(\tau_i^+)}\right) - rI\left(u_i, H(\tau_i^-)\right) \begin{cases} > 0 & \text{if } \tau_i = 0 \\ = 0 & \text{if } \tau_i \in (0, T) \\ < 0 & \text{if } \tau_i = T. \end{cases} \tag{5.34}$$

Appendix 5B Implementation in MATLAB

For the subsequent sections we assume that a solution of (5.1) and time horizon T has already been detected given by $(x^*(\cdot), v_i^*, \tau_i)$, $i = 1, \ldots, N$ with $0 < \tau_1 < \tau_2 < \ldots < \tau_N < T$. In the first section we consider the case where a solution of the canonical system between two adjacent jumps can be found analytically. Therefore the problem can be reduced to a finite number of nonlinear equations, see Section 5.3.2 and 5.3.3.

5B.1 Continuation Algorithm

For the actual implementation in MATLAB a vector x is introduced

$$x = (y(\tau_1^-), y(\tau_1^+), \ldots, y(\tau_N^-), y(\tau_N^+), \tau_1, \ldots, \tau_N)' \tag{5.35a}$$

with

$$y(t) := (x^*(t), \lambda(t)). \tag{5.35b}$$

This vector consists of the left and right side limits of the states and costates at the jumping times and the (interior) jumping times appended at the end. To continue the solution along a parameter value, the initial states or time horizon `MATCONT` is used. Therefore the main `MATCONT` file, where the system is defined, has to be provided.

```
function out = iocmodelDiscrete4matcont
%
% Standard ode file for MATCONT

out{1} = @init;
out{2} = @fun_eval;

out{10}= @interiorjumpfunc;
out{11}= @reachtimehorizon;
out{12}= @jumpingtimesvstimehorizon;
out{13}= @negativetime;
%
% --------------------------------------------------------------------------
function out = fun_eval(t,geny,x0,par,T)
global GIV
aid=GIV.aid;
arcnum=GIV.arcnum;
jid=GIV.jid;
y=geny(GIV.gDVC);
tp=[GIV.IT geny(GIV.JTC).' T];
initres=[];
transres=[];
connecres=[];
dynres=[];
interiorjumpres=[];
for ii=1:arcnum+1
    yLR=y(:,(2*ii-1):2*ii);
    if ii==1
        initres=GIV.IC(tp(ii),yLR,[par,T],aid(1),x0);
    elseif ii==arcnum+1
        transres=GIV.TC(tp(ii),yLR,[par,T],aid(end));
    end
    connecres=[connecres; ...
        GIV.JC(tp(ii),yLR,[par,T],jid(ii))];
```

```
    if arcnum>1 && ii>=2 && ii<=arcnum
        interiorjumpres=[interiorjumpres; ...
            GIV.IJC(tp(ii),yLR,[par,T],aid(ii-1),jid(ii))];
    end
    if ii<=arcnum
        yI=y(:,2*ii:(2*ii+1));
        dynres=[dynres; ...
            GIV.CS(tp(ii:ii+1),yI,[par,T],aid(ii))];
    end
end
out=[initres;transres;connecres;dynres;interiorjumpres];
%----------------------------------------------------------------------
function out=interiorjumpfunc(t,geny,x0,par,T)
global GIV
aid=GIV.aid;
arcnum=GIV.arcnum;
jid=GIV.jid;
y=geny(GIV.gDVC);
yLR=y(:,(2*(arcnum+1)-1):2*(arcnum+1));
if jid(end)
    out=GIV.IJC(T,yLR,[par,T],aid(end),jid(end));
else
    out=1;
end

%----------------------------------------------------------------------
function out=reachtimehorizon(t,geny,x0,par,T)
global GIV

out=GIV.TH-T;
%----------------------------------------------------------------------
function out=jumpingtimesvstimehorizon(t,geny,x0,par,T)
global GIV
tp=[geny(GIV.JTC)];

if isempty(tp)
    out=1;
else
```

```
    out=min(T-tp);
end

function out=negativetime(t,geny,x0,par,T)
global GIV

%----------------------------------------------------------------
out=min([geny(GIV.JTC);T]);
```

Abbreviations

```
GIV=GlobalImpulseVariable
genDynVarCoordinates=gDVC
InitialTime=IT
JumpTimeCoordinates=JTC
TimeHorizon=TH
InteriorJumpCondition=IJC
CanonicalSystem=CS
TransversalityCondition=TC
```

The function fun_eval file defines the ascribing equations. These equations are stated in model specific functions and the function names are defined in the global variable GIV. The fields of the global variable GIV are

arcnum the number of arcs $y(t),\ t \in [\tau_i, \tau_{i+1}],\ i = 0, \ldots, N$ between two adjacent jumping times.

jumparg (jid) an integer vector storing an identifier for each jump. The first and last entry denotes if a jump at the initial or end time occurs. If no jump occurs it is set to zero, otherwise to some integer larger than zero.

InitialTime (IT) stores the initial time t_0.

TimeHorizon (TH) stores the time horizon of the problem T.

CanonicalSystem (CS) function where the canonical system is described.

InteriorJumpCondition (IJC) function for the interior jumping condition (5.7e).

TransversalityCondition (TC) function for the transversality condition (5.7f).

genDynVarCoordinates (gDVC) the matrix of coordinates for the left and right side limits of the states and costates of vector x.

JumpTimeCoordinates (JTC) the coordinates of vector x storing the jumping times.

Further variables used in the listing are

geny variable denoting x of (5.35a).

y matrix, where the column consist of $y(\tau_i^{\pm}),\ i = 1, \dots, N$, as defined in (5.35b).

yLR the left and right side limits $y(\tau^{\pm})$ at a specific jumping time τ.

yI the two column matrix consisting of the right side limit $y(\tau_i^+)$ and the left side limit of the next jumping time $y(\tau_{i+1}^+)$.

tp a vector consisting of the initial time, jumping times and time horizon.

x0 is a vector of the initial states $x(0)$.

par is a vector of the parameter values of the model.

T is the actual time horizon, which need not be equal to the time horizon of the problem stored in GIV.TH.

initres residual of the initial condition.

transres residual of the transversality condition.

connecres residual of the connection between two adjacent arcs.

dynres residual derived from the equations of the canonical system.

interiorjumpres residual derived from the interior jumping conditions.

The user functions used within the `MATCONT` syntax are

interiorjumpfunc returns the value of the interior jumping condition at jumping times. This value is monitored during the continuation process. If it changes sign the necessary jumping condition for an interior jump is satisfied and an interior jump may occur.

reachtimehorizon if the continuation is done with respect to the time horizon this value is monitored to check if the final time horizon is reached.

5B.2. Gradient Algorithm

To solve problem (5.15) numerically the MATLAB function `fmincon` can be used. For that purpose a file describing the objective function and its derivative has to be provided together with a file describing the constraints and the corresponding derivatives. The syntax (as we need it) of the function is

```
x = fmincon(fun,x0,A,b,[],[],lb,ub,nonlcon,opt),
```

where `fun` and `nonlcon` denote the files for the objective function and (nonlinear) constraints, respectively. To apply the gradient algorithm lower and upper bounds of the vector y have to be provided `ub` and `vb`. These bounds should be chosen in a way that the interesting state and control space is covered. If during the calculations the bounds are hit one can increase the bounds to stay in the interior. Furthermore we can assure that the jumping times are ordered and do not exceed the time horizon. These are linear inequalities

$$\tau_1 - \tau_2 \leq 0, \ldots, \tau_{N-1} - \tau_N \leq 0,$$

which can be presented by a matrix inequality of the form $Ay \leq 0$. The vector `x0` is some approximate solution of the problem. For the gradient algorithm the options have at least to consist of

```
opt=optimset('GradObj','on','GradConstr','on');
```

The m-file for the constraints has to return a vector for (nonlinear) equality and inequality constraints and the corresponding derivatives. If the problem does not consist of inequality constraints empty vectors have to be returned.

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